Evaluation and Analysis of Certain Properties of Conformal Gravity in Spherically Symmetric Spacetimes

BY

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DEDICATION

From the very depths of my heart, my consciousness, and my soul; I want to dedicate this work firstly to Almighty Allah, the essence of my life, my first love and the source of all the knowledge and energy that I have poured into this work, and secondly, to my parents whose unimaginable support has made it possible for me to get to this height in life. I lastly also dedicate this work to my thesis advisor, Dr. Thamer Al-Aithan, co-advisor, Dr. Ashfaque Bokhari and all members of the thesis committee which includes Dr. Fatah-Zouhir Khiari, Dr. James Wheeler, and Dr. Hocine Bahlouli. This work's beauty lies on your hands.

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ABSTRACT

General theory of relativity and Weyl (Conformal) gravity are two different ways of explaining the same phenomenon; gravity. Known solutions of Einstein's field equations have been studied extensively from the view point of classical relativity. Studies have also been carried out to understand if properties of Einstein gravity are also applicable to its Weyl version. In this thesis, we review as a comparison, some concepts as viewed from both the classical and the Weyl context in spherically symmetric spacetime metrics. Keeping in view fact that Birkhoff's theorem is one of the pivotal results in Relativity, namely, any spherically symmetric solution of the Einstein-Maxwell field equations must be static and asymptotically flat. We focus on giving a review of Birkhoff's theorem and showing how it holds for both general relativity and conformal gravity. We also discuss in detail the solution of the conformally transformed spherically symmetric metric and obtain the energy equation, plot its potential curves and a brief discussion of these results is given. We also showed that the Noether current for Weyl gravity vanishes identically.

ملخص البحث

إن النسبية العامة (General Gravity) وقانون جاذبية وايل الامتثالي (Weyl Conformal Gravity) طريقتان مختلفتان لشرح نفس الظاهرة: الجاذبية، وقد تم دراسة الحلول المعروفة من معادلات ميدان آينشتاين (Einstein's Field Equation) على نطاق واسع ووُجِدَت في كثير من الأحيان اتصافها بخصائص معينة؛ والباحثون في هذا المجال قد واصلوا في العمل لمعرفة ما إذا كان هذه الخصائص الفريدة لجاذبية آينشتاين تنطبق أيضا على نسخة وايل.

وفي هذا العمل، نقارن بين بعض المفاهيم كما يُنْظَر إليها من كلتا النظريتين باستخدام نظام القياس المتماثل المتري الكروي (spherically symmetric metric). من خصائص جاذبية آينشتاين أن أي حل ناتج من المتماثل الكروي لمعادلات الميدان الفراغي يجب أن يكون ثابتا ومسطَّحا مقاربا (نظرية Birkhoff). وبعبارة أخرى، إن الحل الخارجي الذي يصوّر الزمكنة خارج الجسم المنجذب من غير دوران يجب أن يقدّمه أسلوب شوارزشيلد المتري. فنبدأ باستعراض هذه النظرية في كل من النظريتين النسبية العامة (General Gravity) وقانون جاذبية وايل الامتثالي (Weyl Conformal Gravity) ثم دققنا النظر في الحل الناتج من جاذبية فايل باستخدام نظام القياس المتماثل المتري الكروي (spherically symmetric metric) ثم وجدنا معادلة الطاقة و الرسوم البياني لها و تحدثنا عن النتائجز و أيضا بينا أن تيار النيثار (Noether Current) الناتج من قانون جاذبية وايل الامتثالي (Weyl Conformal Gravity) يختفي تماما.

CHAPTER ONE

INTRODUCTION

1.1 GENERAL RELATIVITY

After Einstein developed special relativity, he knew there was a greater task ahead. He must also develop a theory to fix Newton's laws of gravity. He then let go the notion of forces as the source of gravitational attraction and introduced the concept of curved spacetime and motion along geodesics. This is a completely different way of explaining the same thing and turned out to be the most reliable and effective. This is general relativity and is based on the assumptions of general coordinate covariance and the equivalence principle.

The idea of general coordinate invariance requires the concept of tensor calculus. Tensors are the building blocks of the theory of relativity. A tensor is an object that transforms linearly and homogenously under coordinate transformations. Therefore, if a tensor equation holds in one set of coordinates, it holds in every set of coordinates.

1.1.1 The Metric

The metric, also known as the metric tensor, is a symmetric tensor that forms an intrinsic part of general relativity. At each point of a Riemannian manifold, and in the case of general relativity at each event of spacetime, there exists this geometrical object called the metric tensor. In differential geometry, a (smooth) Riemannian manifold or (smooth) Riemannian space (M, g) is a real smooth manifold M equipped with an inner product on the tangent space at each point that varies smoothly from point to point in the sense that if X and Y are vector fields on M, then it is a smooth function. There are two ways to think of the metric tensor. In its simplest form in an infinite dimensional tangent space, we can define it as an inner product of vectors thereby writing a mathematical representation of the metric as follows,

$$\boldsymbol{g}(\boldsymbol{u},\boldsymbol{v}) = g_{\alpha\beta}u^{\alpha}v^{\beta} = g_{11}u^1v^1 + g_{12}u^1v^2 + g_{21}u^2v^1 + \cdots \tag{1}$$

where u and v are any arbitrary vectors, in 4-dimensional spacetime.

We can also define the metric as the infinitesimal distance between separated points or vector length. In 4-dimensional flat spacetime,

$$g_{00} = -1, g_{ok} = 0, g_{jk} = \delta_{jk} \tag{2}$$

In matrix form, the above expression is written as a diagonal matrix, having all off diagonal elements equal zero, in the form,

$$g_{\alpha\beta} = \eta_{\alpha\beta} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \tag{3}$$

and this gives the length of any 4-vector in a four-dimensional spacetime. Generally, the metric is an arbitrary, symmetric, and non-degenerate matrix which defines the length of an infinitesimal line vector *ds* via;

$$ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta} \tag{4}$$

All the geometric and causal structure of spacetime used to define notions such as time, distance, volume, curvature, angle, and separating the past and future are captured in the metric.

Other useful properties of the metric include the following:

(a) It can raise or lower indices:

$$T^{\alpha\beta} g_{\mu\nu} = T^{\beta}{}_{\mu} \tag{5}$$

(b) It transforms as a tensor:

$$g'_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\upsilon}}{\partial x'^{\beta}} g_{\mu\nu} \tag{6}$$

(c) $g^{\mu\nu} g_{\mu\nu} = 4$ (7)

(d) It is a symmetric tensor ($g_{\mu\nu} = g_{\upsilon\mu}$), with 16 components reduced to 10 due to symmetry.

Tensors with indices up, like $g^{\mu\nu}$, are called contravariant tensors while those with indices down, like $g_{\mu\nu}$, are called covariant tensors. Covariant and contravariant tensors are notations used to define elements like vectors from a linear vector space and a dual space respectively in which an inner product can be defined.

1.1.2 The Connection

There is an important difference between the Lorentz transformations and the general coordinate transformations. In the case of Lorentz transformations we are able to differentiate vectors in the usual way to get other vectors. Thus, if we wanted the derivatives of a vector v^{α}, we could simply compute them all and have a type-$\binom{1}{1}$ tensor,

$$T^{\alpha}{}_{\beta} = \frac{\partial v^{\alpha}}{\partial x^{\beta}}$$

Then $T^{\alpha}{}_{\beta}$ is a tensor, because in any other Lorentz frame,

$$\bar{T}^{\alpha}{}_{\beta} = \Lambda^{\upsilon}{}_{\beta}\frac{\partial}{\partial x^{\upsilon}}\left(\Lambda^{\alpha}{}_{\mu}v^{\mu}\right) = \Lambda^{\upsilon}{}_{\beta}\Lambda^{\alpha}{}_{\mu}\frac{\partial}{\partial x^{\upsilon}}(v^{\mu}) = \Lambda^{\upsilon}{}_{\beta}\Lambda^{\alpha}{}_{\mu}T^{\mu}{}_{\upsilon}$$

This happens only because $\Lambda^{\alpha}{}_{\beta}$ is constant. However, if a vector v^{α} transforms with a change of coordinates such as, $v^{\alpha} = \frac{\partial y^{\alpha}}{\partial x^{\beta}}v^{\beta}$, then its derivative is not a tensor. Instead,

$$\frac{\partial}{\partial y^{\beta}}(v^{\alpha}) = \frac{\partial}{\partial y^{\beta}}\left(\frac{\partial y^{\alpha}}{\partial x^{\mu}}v^{\mu}\right)$$

$$= \left(\frac{\partial x^{\upsilon}}{\partial y^{\beta}}\frac{\partial}{\partial x^{\upsilon}}\right)\left(\frac{\partial y^{\alpha}}{\partial x^{\mu}}v^{\mu}\right)$$

$$= \frac{\partial x^{\upsilon}}{\partial y^{\beta}}\frac{\partial y^{\alpha}}{\partial x^{\mu}}\frac{\partial}{\partial x^{\upsilon}}v^{\mu} + \frac{\partial x^{\upsilon}}{\partial y^{\beta}}\frac{\partial^{2}y^{\alpha}}{\partial x^{\upsilon}\partial x^{\mu}}v^{\mu}$$

$$= \frac{\partial x^{\upsilon}}{\partial y^{\beta}}\frac{\partial y^{\alpha}}{\partial x^{\mu}}\frac{\partial}{\partial x^{\upsilon}}v^{\mu} + \frac{\partial x^{\upsilon}}{\partial y^{\beta}}\left(\frac{\partial y^{\alpha}}{\partial x^{\mu}}\frac{\partial x^{\mu}}{\partial y^{\sigma}}\right)\frac{\partial^{2}y^{\sigma}}{\partial x^{\upsilon}\partial x^{\rho}}v^{\rho}$$

$$= \frac{\partial x^{\upsilon}}{\partial y^{\beta}}\frac{\partial y^{\alpha}}{\partial x^{\mu}}\left(\frac{\partial}{\partial x^{\upsilon}}v^{\mu} + \frac{\partial x^{\mu}}{\partial y^{\sigma}}\frac{\partial^{2}y^{\sigma}}{\partial x^{\upsilon}\partial x^{\rho}}v^{\rho}\right)$$

which is not a tensor.

We define the *covariant derivative* in such a way as to correct this problem and produce a tensor. The idea is to add another term to the partial derivative, and let the extra term change in just the right way to cancel the extra, inhomogeneous part. Define

$$D_{\beta}v^{\alpha} \equiv \partial_{\beta}v^{\alpha} + v^{\mu}\Gamma^{\alpha}{}_{\mu\beta}$$

and require $\Gamma^{\beta}{}_{\mu\alpha}$ to transform so that $D_{\alpha}v^{\beta}$ transforms as a tensor when we change coordinates. That is we require the covariance condition,

$$\overline{D}_{\beta}v^{\alpha} = \frac{\partial x^{\upsilon}}{\partial y^{\beta}}\frac{\partial y^{\alpha}}{\partial x^{\mu}}D_{\upsilon}v^{\upsilon} \tag{8}$$

where

$$D_{\beta}v^{\alpha} = \frac{\partial}{\partial y^{\beta}}v^{\alpha} + v^{\mu}\Gamma^{\alpha}{}_{\mu\beta}$$

$$v^{\alpha} = \frac{\partial y^{\alpha}}{\partial x^{\mu}}v^{\mu}$$

The symbol $\Gamma^{\alpha}{}_{\mu\beta}$ is called the Christoffel symbol.

Substituting into (8),

$$\overline{D}_{\beta}v^{\alpha} = \frac{\partial x^{\upsilon}}{\partial y^{\beta}}\frac{\partial y^{\alpha}}{\partial x^{\mu}}D_{\upsilon}v^{\upsilon}$$

$$= \frac{\partial x^{\upsilon}}{\partial y^{\beta}}\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial y^{\alpha}}{\partial x^{\mu}}v^{\mu}\right) + \left(\frac{\partial y^{\nu}}{\partial x^{\mu}}v^{\mu}\right)\overline{\Gamma}^{\alpha}{}_{\nu\beta} = \frac{\partial x^{\upsilon}}{\partial y^{\beta}}\frac{\partial y^{\alpha}}{\partial x^{\mu}}\left(\frac{\partial v^{\mu}}{\partial x^{\nu}} + v^{\sigma}\Gamma^{\mu}{}_{\sigma\nu}\right)$$

$$= \frac{\partial x^{\upsilon}}{\partial y^{\alpha}}\frac{\partial y^{\beta}}{\partial x^{\mu}}\left(\frac{\partial}{\partial x^{\upsilon}}v^{\mu} + \frac{\partial x^{\mu}}{\partial y^{\sigma}}\frac{\partial^{2}y^{\sigma}}{\partial x^{\upsilon}\partial x^{\rho}}v^{\rho}\right) + \left(\frac{\partial y^{\nu}}{\partial x^{\mu}}v^{\mu}\right)\overline{\Gamma}^{\alpha}{}_{\nu\beta} = \frac{\partial x^{\upsilon}}{\partial y^{\beta}}\frac{\partial y^{\alpha}}{\partial x^{\mu}}\left(\frac{\partial v^{\mu}}{\partial x^{\nu}} + v^{\sigma}\Gamma^{\mu}{}_{\sigma\nu}\right)$$

$$= \frac{\partial x^{\upsilon}}{\partial y^{\alpha}}\frac{\partial y^{\beta}}{\partial x^{\mu}}\left(\frac{\partial x^{\mu}}{\partial y^{\sigma}}\frac{\partial^{2}y^{\sigma}}{\partial x^{\upsilon}\partial x^{\rho}}v^{\rho}\right) + \left(\frac{\partial y^{\nu}}{\partial x^{\rho}}v^{\rho}\right)\overline{\Gamma}^{\alpha}{}_{\nu\beta} = \frac{\partial x^{\upsilon}}{\partial y^{\beta}}\frac{\partial y^{\alpha}}{\partial x^{\mu}}\left(v^{\rho}\Gamma^{\mu}{}_{\rho\nu}\right)$$

This must hold for every vector, v^{ρ}, so,

$$\frac{\partial x^{\upsilon}}{\partial y^{\alpha}}\frac{\partial y^{\beta}}{\partial x^{\mu}}\left(\frac{\partial x^{\mu}}{\partial y^{\sigma}}\frac{\partial^{2}y^{\sigma}}{\partial x^{\upsilon}\partial x^{\rho}}v^{\rho}\right) + \left(\frac{\partial y^{\nu}}{\partial x^{\rho}}v^{\rho}\right)\overline{\Gamma}^{\alpha}{}_{\nu\beta} = \frac{\partial x^{\upsilon}}{\partial y^{\beta}}\frac{\partial y^{\alpha}}{\partial x^{\mu}}\left(v^{\rho}\Gamma^{\mu}{}_{\rho\nu}\right)$$

$$\left(\frac{\partial y^{\nu}}{\partial x^{\rho}} v^{\rho}\right)\overline{\Gamma}^{\alpha}{}_{\nu\beta} = \frac{\partial x^{\upsilon}}{\partial y^{\beta}}\frac{\partial y^{\alpha}}{\partial x^{\mu}}\left(v^{\rho}\Gamma^{\mu}{}_{\rho\nu}\right) - \frac{\partial x^{\upsilon}}{\partial y^{\alpha}}\frac{\partial y^{\beta}}{\partial x^{\mu}}\left(\frac{\partial x^{\mu}}{\partial y^{\sigma}}\frac{\partial^2 y^{\sigma}}{\partial x^{\upsilon}\partial x^{\rho}} v^{\rho}\right)$$

$$\frac{\partial x^{\rho}}{\partial y^{\lambda}}\left(\frac{\partial y^{\nu}}{\partial x^{\rho}} v^{\rho}\right)\overline{\Gamma}^{\alpha}{}_{\nu\beta} = \frac{\partial x^{\rho}}{\partial y^{\lambda}}\left[\frac{\partial x^{\upsilon}}{\partial y^{\beta}}\frac{\partial y^{\alpha}}{\partial x^{\mu}}\left(v^{\rho}\Gamma^{\mu}{}_{\rho\nu}\right) - \frac{\partial x^{\upsilon}}{\partial y^{\alpha}}\frac{\partial y^{\beta}}{\partial x^{\mu}}\left(\frac{\partial x^{\mu}}{\partial y^{\sigma}}\frac{\partial^2 y^{\sigma}}{\partial x^{\upsilon}\partial x^{\rho}} v^{\rho}\right)\right]$$

$$\overline{\Gamma}^{\alpha}{}_{\lambda\beta} = \frac{\partial x^{\rho}}{\partial y^{\lambda}}\frac{\partial x^{\upsilon}}{\partial y^{\beta}}\frac{\partial y^{\alpha}}{\partial x^{\mu}}\left(\Gamma^{\mu}{}_{\rho\nu} - \frac{\partial x^{\mu}}{\partial y^{\sigma}}\frac{\partial^2 y^{\sigma}}{\partial x^{\upsilon}\partial x^{\rho}}\right)$$

In terms of the metric we can show from the covariant constancy of the metric that,

$$\Gamma^{\beta}{}_{\mu\nu} = \frac{1}{2} g^{\beta\alpha}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right) \tag{9}$$

1.1.3 The Riemann Tensor

When two identical vectors are parallel transported along different curves from a single point to another, the vectors' orientations do not emerge the same as they started. Riemann used this fact to understand the concept of curvature in spacetime and came up with what we know now as the Riemann tensor. The Riemann tensor is that tensor which tells the difference between a flat space and a curved space.

If A^{α} is a vector at a point P along a curve, and U^{β} is the tangent to the curve, then, the condition for the parallel transport of A^{α} along the curve requires that we solve for A^{α} when the directional derivative vanishes. i.e. $U^{\alpha} D_{\alpha} A^{\beta} = 0$.

Consider the figure below;

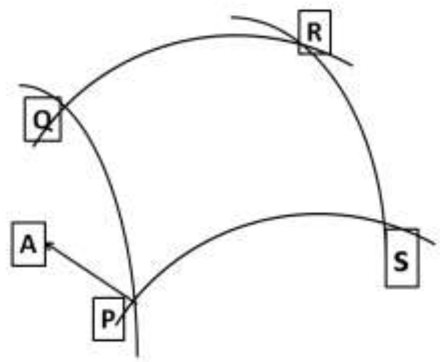


Figure 1.1: Desrcription of Parallel Transport of a Tangent vector A along a closed path PQRS

If we parallel transport an arbitrary vector, **A** from P to R along two different paths PQR and PSR we get the following:

From P to Q,

$$A^{\alpha}(Q) = A^{\alpha}(P) - \Gamma^{\alpha}{}_{\beta\mu}(P)A^{\beta}(P)d\lambda$$

and from Q to R,

$$A^{\alpha} - \Gamma^{\alpha}{}_{\beta\mu}(P')A^{\beta}d\lambda - \Gamma^{\alpha}{}_{\sigma\upsilon}(Q')\left[A^{\sigma} - \Gamma^{\sigma}{}_{\beta\mu}(P')A^{\beta}d\lambda\right]dS$$

In terms of P',

$$\Gamma^{\alpha}{}_{\sigma\upsilon}(Q') \cong \Gamma^{\alpha}{}_{\sigma\upsilon}(P') + \Gamma^{\alpha}{}_{\sigma\upsilon,\mu}(P')d\lambda$$

$$\therefore A^{\alpha} - \Gamma^{\alpha}{}_{\beta\mu}(P')A^{\beta}d\lambda - \Gamma^{\alpha}{}_{\sigma\upsilon}(P')A^{\sigma}dS + \Gamma^{\alpha}{}_{\sigma\upsilon}(P')\Gamma^{\sigma}{}_{\beta\mu}(P')A^{\beta}d\lambda dS - \Gamma^{\alpha}{}_{\sigma\upsilon,\beta}(P')A^{\sigma}d\lambda dS$$

Then similarly along the path P'PQ we have,

$$A^{\alpha} - \Gamma^{\alpha}{}_{\beta\mu}(P')A^{\beta}dS - \Gamma^{\alpha}{}_{\sigma\upsilon}(P')A^{\sigma}d\lambda + \Gamma^{\alpha}{}_{\sigma\upsilon}(P')\Gamma^{\sigma}{}_{\beta\mu}(P')A^{\beta}d\lambda dS - \Gamma^{\alpha}{}_{\sigma\upsilon,\mu}(P')A^{\sigma}d\lambda dS$$

$$\therefore \Delta A^{\alpha} = \left\{-\Gamma^{\alpha}{}_{\beta\gamma,\nu} + \Gamma^{\alpha}{}_{\beta\nu,\mu} + \Gamma^{\alpha}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\sigma\upsilon}\Gamma^{\sigma}{}_{\beta\mu}\right\}A^{\beta}d\lambda dS$$

$$= R^{\alpha}{}_{\beta\mu\nu}A^{\beta}d\lambda dS$$

where $-\Gamma^{\alpha}{}_{\beta\gamma,\nu} + \Gamma^{\alpha}{}_{\beta\nu,\mu} + \Gamma^{\alpha}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\sigma\upsilon}\Gamma^{\sigma}{}_{\beta\mu} = R^{\alpha}{}_{\beta\mu\nu}$ called the Riemann Tensor.

It has 256 components and in general can be written in the form,

$R^{\lambda}_{\mu\nu\kappa} = The\ Rieman\ tensor$

$$= \frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial\Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta} = \Gamma^{\lambda}_{\mu\nu,\kappa} - \Gamma^{\lambda}_{\mu\kappa,\nu} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta} \qquad (10)$$

The Riemann tensor possesses some unique symmetry properties which are

$R_{abcd} = -R_{abdc} = -R_{bacd} = R_{cdab}$

$R_{abcd} + R_{adbc} + R_{acdb} = 0$

These symmetries reduce the components of the Riemann tensor to 20 independent components.

1.1.4 The Ricci Tensor

The Riemann tensor can be decoupled into a symmetric and traceless part, each with ten independent components respectively. The symmetric part of the Riemann tensor is what we refer to as the Ricci tensor, while the traceless part of the Riemann tensor is the Weyl tensor.

Mathematically, we write the Ricci tensor as $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ where,

$$R_{\mu\kappa} = R^{\lambda}_{\mu\lambda\kappa} = \Gamma^{\lambda}_{\mu\lambda,\kappa} - \Gamma^{\lambda}_{\mu\kappa,\lambda} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\lambda\eta}$$

$$= \frac{\partial\Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\kappa}} - \frac{\partial\Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\lambda\eta} \qquad (11)$$

which is symmetric, i.e.,

$$R_{\mu\nu} = R_{\nu\mu} \qquad (12)$$

1.1.5 Geodesics

From the parallel transport phenomenon we define a covariant derivative of a 4-dimensional vector;

$$\frac{d\boldsymbol{V}}{d\lambda} = \frac{\partial V^{\mu}}{\partial\lambda} + \Gamma^{\mu}{}_{\alpha\beta} V^{\alpha} \frac{\partial x^{\beta}}{\partial\lambda} = 0 \qquad (13)$$

If we on the other hand, instead of parallel transporting an arbitrary vector, we decide to parallel transport the tangent vector, we get a similar equation but in terms of the tangent.

$$\frac{d^2 x^{\mu}}{d\lambda^2} + \Gamma^{\mu}{}_{\alpha\beta} \frac{\partial x^{\alpha}}{\partial\lambda} \frac{\partial x^{\beta}}{\partial\lambda} = 0 \qquad (14)$$

which is called the geodesic equation. For a plane, it is a line. For a sphere, it is a great circle. The geodesics do not change as we change from one coordinate system to another. The equations can also be derived from the principle of least action.

In general, a geodesic is the path a particle will follow in spacetime if it is not accelerating. In other words it is a locally length-minimizing curve.

Consider a curve, $x^\alpha(\lambda)$ in an arbitrary (possibly curved) spacetime, with the proper interval given by:

$$d\tau^2 = -g_{\alpha\beta}dx^\alpha dx^\beta \tag{15}$$

Then the 4-velocity along the curve is given by:

$$u^\alpha = \frac{dx^\alpha}{d\tau} \tag{16}$$

and in an arbitrary parameterization, the tangent is $t^\alpha = \frac{dx^\alpha}{d\lambda}$. Then the proper time (or length) along the curve is given by:

$$\tau = \int_0^\tau -g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}d\lambda \tag{17}$$

A curve of extremal proper length is the geodesic. We may find an expression for geodesics by finding the equation for the extrema of τ,

$$0 = \delta\tau$$

$$= \delta\int_0^\tau -g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}d\lambda$$

$$= -\int_0^\tau \frac{1}{2\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}\delta\left(g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\right)d\lambda$$

$$= -\int_0^\tau \frac{1}{2\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}\left(\delta g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} + g_{\alpha\beta}\frac{d\delta x^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} + g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{d\delta x^\beta}{d\lambda}\right)d\lambda$$

$$= -\int_0^\tau \frac{1}{2\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}\left(g_{\alpha\beta,\mu}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\right)\delta x^\mu d\lambda$$

$$+\int_0^\tau \frac{d}{d\lambda}\left(\frac{1}{2\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}\right)\left(g_{\alpha\beta}\frac{dx^\beta}{d\lambda}\right)\delta x^\alpha d\lambda$$

$$+\int_0^\tau \frac{d}{d\lambda}\left(\frac{1}{2\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}\right)\left(g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\right)\delta x^\beta d\lambda$$

Now we choose the parameter λ to be proper time (length) so that,

$$g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = -c^2 = -1$$

Then we have:

$$0 = \frac{1}{2}\int_0^\tau \left[\left(g_{\alpha\beta,\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right)\delta x^\mu + \frac{d}{d\tau}\left(g_{\alpha\beta}\frac{dx^\beta}{d\tau}\right)\delta x^\alpha + \frac{d}{d\tau}\left(g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\right)\delta x^\beta\right]d\lambda$$

$$= \frac{1}{2}\int_0^\tau \left[(-g_{\alpha\beta,\mu}u^\alpha u^\beta)\delta x^\mu + \frac{d}{d\tau}(g_{\alpha\beta}u^\beta)\delta x^\alpha + \frac{d}{d\tau}(g_{\alpha\beta}u^\alpha)\delta x^\beta\right]d\lambda$$

$$= \frac{1}{2}\int_0^\tau \left[(-g_{\alpha\beta,\mu}u^\alpha u^\beta)\delta x^\mu + \left(g_{\alpha\beta,\nu}u^\beta\frac{dx^\nu}{d\tau} + g_{\alpha\beta}\frac{du^\beta}{d\tau}\right)\delta x^\alpha\right.$$

$$\left. + \left(g_{\alpha\beta,\nu}u^\alpha\frac{dx^\nu}{d\tau} + g_{\alpha\beta}\frac{du^\alpha}{d\tau}\right)\delta x^\beta\right]d\lambda$$

$$= \frac{1}{2}\int_0^{\tau}\left[\left(-g_{\alpha\beta,\mu}u^{\alpha}u^{\beta}\right) + \left(g_{\alpha\beta,\nu}u^{\nu}u^{\beta} + g_{\alpha\beta}\frac{du^{\beta}}{d\tau}\right) + \left(g_{\alpha\beta,\nu}u^{\alpha}u^{\nu} + g_{\alpha\beta}\frac{du^{\alpha}}{d\tau}\right)\right]\delta x^{\mu}d\lambda$$

$$= \frac{1}{2}\int_0^{\tau}\left[\left(g_{\mu\beta,\nu}u^{\nu}u^{\beta} + g_{\alpha\mu,\nu}u^{\nu}u^{\alpha} - g_{\alpha\beta,\mu}u^{\alpha}u^{\beta} + g_{\mu\beta}\frac{du^{\beta}}{d\tau} + g_{\alpha\mu}\frac{du^{\alpha}}{d\tau}\right)\right]\delta x^{\mu}d\lambda$$

The equation for the geodesic is therefore,

$$0 = \frac{1}{2}\left(g_{\mu\beta,\nu}u^{\nu}u^{\beta} + g_{\alpha\mu,\nu}u^{\nu}u^{\alpha} - g_{\alpha\beta,\mu}u^{\alpha}u^{\beta} + g_{\mu\beta}\frac{du^{\beta}}{d\tau} + g_{\alpha\mu}\frac{du^{\alpha}}{d\tau}\right)$$

$$= \frac{1}{2}\left(g_{\mu\beta,\alpha} + g_{\alpha\mu,\beta} - g_{\alpha\beta,\mu}\right)u^{\alpha}u^{\beta} + g_{\mu\beta}\frac{du^{\beta}}{d\tau}$$

$$= g^{\mu\nu}g_{\mu\beta}\frac{du^{\beta}}{d\tau} + \frac{1}{2}g^{\mu\nu}\left(g_{\mu\beta,\alpha} + g_{\alpha\mu,\beta} - g_{\alpha\beta,\mu}\right)u^{\alpha}u^{\beta}$$

$$= \frac{du^{\nu}}{d\tau} + \Gamma^{\nu}{}_{\alpha\beta}u^{\alpha}u^{\beta}$$

But this is just:

$$\frac{du^{\nu}}{d\tau} + \Gamma^{\nu}{}_{\alpha\beta}u^{\alpha}u^{\beta} = u^{\mu}\left(\frac{du^{\nu}}{dx^{\mu}} + u^{\alpha}\Gamma^{\nu}{}_{\alpha\mu}\right) = u^{\mu}D_{\mu}u^{\nu}$$

which implies,

$$u^{\mu}D_{\mu}u^{\nu} = \frac{du^{\nu}}{d\tau} + \Gamma^{\nu}{}_{\alpha\beta}u^{\alpha}u^{\beta} = 0 \tag{18}$$

1.1.6 The Einstein-Hilbert Action and Einstein's Field Equations

In 1915, Hilbert proposed the action that yields the Einstein's field equations as follows

$$I_E = -\frac{1}{16\pi G}\int d^4\, x\sqrt{g}.R \tag{19}$$

This action describes gravitational fields in the absence of matter, with $R$is the Ricci Scalar, $g_{\mu\nu}(x)$ a metric, $R_{\mu\nu}$ the Ricci tensor, and g is the $-det g_{\mu\nu}(x)$.

Varying (12), and requiring that $\delta I_E = 0$, we find $\delta\sqrt{g}.R$

$$\delta\sqrt{g}.R = \delta\left(\sqrt{g}.g^{\mu\nu}R_{\mu\nu}\right) \tag{20}$$

$$= \sqrt{g}.R_{\mu\nu}\delta g^{\mu\nu} + \delta\sqrt{g}.R_{\mu\nu}g^{\mu\nu} + \sqrt{g}.\delta R_{\mu\nu}g^{\mu\nu} \tag{21}$$

From the last term in (21), we get,

$$\delta R_{\mu\nu} = \frac{\partial\delta\Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\kappa}} - \frac{\partial\delta\Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\lambda}} + \delta(\Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\kappa\eta}) - \delta(\Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\lambda\eta}) \tag{22}$$

$$\delta\Gamma^{\lambda}_{\mu\nu} = -g^{\lambda\rho}\delta g_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} + \cdots \tag{23}$$

Using the Palatini identity for the last term in (21),

$$\delta R_{\mu\nu} = (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu} - (\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} \tag{24}$$

The above equation states that the variation of the Ricci tensor with respect to the metric may be expressed in terms of total derivatives.

Geodesic coordinates are often used to derive many tensor identities, where an arbitrary point P is chosen at which, $\Gamma^{a}_{bc} = 0$. Then, at this point P, covariant derivatives are reduced to ordinary derivatives. Thus if we rewrite the Riemann tensor as,

$$R^{a}{}_{bcd} = \Gamma^{a}{}_{bd,c} - \Gamma^{a}{}_{bc,d} + \Gamma^{e}_{bd}\Gamma^{a}_{ec} - \Gamma^{e}_{bc}\Gamma^{a}_{ed} \tag{25}$$

the equation reduces to:

$$R^{a}{}_{bcd} = \Gamma^{a}{}_{bd,c} - \Gamma^{a}{}_{bc,d} \tag{26}$$

We now contemplate a variation of the connection $\Gamma^{a}{}_{bc}$ to a new connection $\bar{\Gamma}^{a}{}_{bc}$:

$$\Gamma^{a}{}_{bc} \to \bar{\Gamma}^{a}{}_{bc} = \Gamma^{a}{}_{bc} + \delta\Gamma^{a}{}_{bc} \tag{27}$$

Then $\delta\Gamma^{a}{}_{bc}$ being the difference of two connections is a tensor of type (1, 2). This variation results in a change in the Riemann tensor:

$$R^{a}{}_{bcd} \to \bar{\mathrm{R}}^{a}{}_{bcd} = R^{a}{}_{bcd} + \delta R^{a}{}_{bcd} \tag{28}$$

where in geodesic coordinates,

$$\delta R^{a}{}_{bcd} = \partial_{c}(\delta\Gamma^{a}{}_{bd}) - \partial_{d}(\delta\Gamma^{a}{}_{bc}) = \nabla_{c}(\delta\Gamma^{a}{}_{bd}) - \nabla_{d}(\delta\Gamma^{a}{}_{bc}) \tag{29}$$

since partial derivative commutes with variation and is equivalent to covariant derivative in geodesic coordinates. Now since $\delta R^{a}{}_{bcd}$, being the difference of two tensors, is a tensor, and the quantities on the right-hand side of the last equation are tensors, we may use the fundamental property of tensors; (if a tensor equation holds in one coordinate system it must hold in all coordinate systems), to declare the **Palatini equation** (30) at the point P.

$$\delta R^{a}{}_{bcd} = \nabla_{c}(\delta\Gamma^{a}{}_{bd}) - \nabla_{d}(\delta\Gamma^{a}{}_{bc}) \tag{30}$$

Since P is an arbitrary point, the result holds quite generally and contraction on '*a'* and '*c'* gives the useful result called the Palatini identity in (24). Using this identity to simplify the last term in (21) we get,

$$\sqrt{g}.\, g^{\mu\nu}\delta R_{\mu\nu} = \sqrt{g}.\left[(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu} - (g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda}\right]$$

Using the definition for covariant divergence,

$$V^{\mu}{}_{;\mu} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}\sqrt{g}.\, V^{\mu},$$

gives,

$$\sqrt{g}.\, g^{\mu\nu}\delta R_{\mu\nu} = \frac{\partial}{\partial x^{\nu}}(\sqrt{g}.\, g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\lambda}) - \frac{\partial}{\partial x^{\lambda}}(\sqrt{g}.\, g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu}) \tag{31}$$

This depicts a total derivative whose integral can be written as a boundary term, and at infinity, we take the variation of the fields to vanish, i.e. ,

$$\int \sqrt{g}.\, g^{\mu\nu}\delta R_{\mu\nu}\, d^4x = 0 \tag{32}$$

It may be shown that,

$$\delta\sqrt{g} = \frac{1}{2}\sqrt{g}.\, g^{\mu\nu}\delta g_{\mu\nu} \tag{33}$$

and we have;

$$\delta g^{\mu\nu} = -g^{\mu\sigma}g^{\nu\lambda}\delta g_{\sigma\lambda} = -\delta g_{\mu\nu} \tag{34}$$

Substituting (33) and (34) into (21) and then (21) into (19), we get;

$$\delta I_E = 0 \Rightarrow -\frac{1}{16\pi G}\int d^4\,x\sqrt{g}.\left(-R_{\mu\nu} + \frac{1}{2}g^{\mu\nu}R_{\mu\nu}\right)\delta g_{\mu\nu} = 0$$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{35}$$

We define the energy-momentum tensor for a material system described by an action I_M as the "functional derivative" of $I_M$with respect to $g_{\mu\nu}$. That is, we imagine $\delta g_{\mu\nu}(x)$to be subject to an infinitesimal variation,

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + \delta g_{\mu\nu} \tag{36}$$

where $\delta g_{\mu\nu}$ is arbitrary but required to vanish as $\left|x^\lambda\right| \longrightarrow \infty$. The action I_M will not be stationary with respect to this variation, because for the moment we are regarding $g_{\mu\nu}(x)$ as an external field and not as a dynamical variable. Hence, δI_M will be some linear functional of the infinitesimal $\delta g_{\mu\nu}(x)$, and therefore takes the form,

$$\delta I_M = \frac{1}{2}\int d^4\,x\sqrt{g(x)}.T^{\mu\nu}(x)\delta g_{\mu\nu}(x) \tag{37}$$

where the coefficient $T^{\mu\nu}(x)$ is defined to be the energy-momentum tensor of this system.

Now, we can represent the total action as follows:

$$I = I_M + I_E \;\;\Rightarrow\;\;\;\; \delta I = \delta I_M + \delta I_E \tag{38}$$

Using (35) and (37), we deduce a total action that is stationary with respect to arbitrary variations in $g_{\mu\nu}$ if and only if,

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + 8\pi G T^{\mu\nu} = 0 \tag{39}$$

Equation (39) is the Einstein field equations in the presence of matter

The field equations carry a lot of information that cannot be overemphasized. For instance, in 1915, Einstein used these equations to correctly compute the anomalous precession of the orbit of Mercury and also the deflection of starlight by the Sun's gravitational field. In the weak field limit, this theory includes Newton's inverse-square force law , and also equations governing the big bang cosmology. Some major applications and predictions inherent in these equations include tidal forces, gravitational waves, spacetime curvature, big bang, expansion of the universe, and GPS.

1.2 CONFORMAL GRAVITY

Conformal gravity refers to gravity theories that are invariant under conformal or Weyl transformations. This idea was introduced by Weyl in 1918, when he proposed a unification of gravitation and electromagnetism based on the invariance of physics with respect to a conformal (or scale) transformation of the metric tensor [2].

Weyl's theory, introduced a non-Riemannian geometry which relied upon a Lagrangian that was invariant with respect to a local rescaling of the metric tensor.

1.2.1 Conformal Transformation

A conformal transformation may be defined in several ways. The simplest definition may be to say that conformal transformations are mappings of space that preserve angles. Such a transformation will take a triangle to a similar triangle, but the new triangle may be in a different place (translated), or rotated, or a different size.

Since sines and cosines are given by ratios of the lengths of sides of triangles, preserving angles is equivalent to preserving ratios of lengths. This is an equivalent definition, and one useful for physics, because any measurement is ultimately a comparison of two magnitudes.

In flat space-time, it turns out that a transformation preserves ratios of lengths if and only if it preserves light cones.

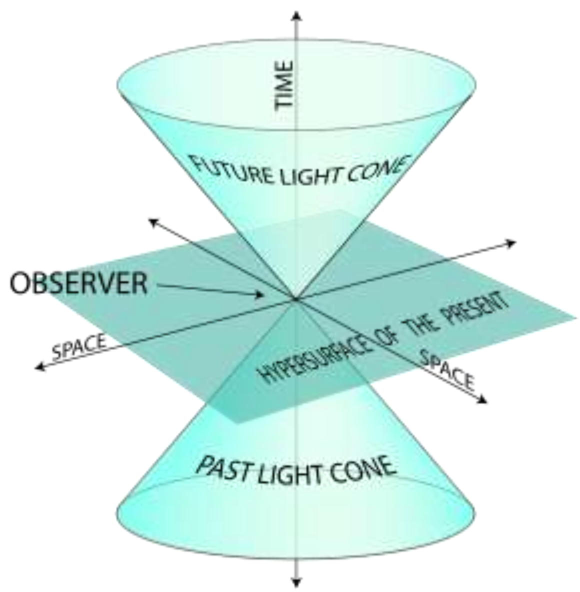


Figure 1.2: the light cone showing the past, present, and future in spacetime

In general relativity and related theories of curved spacetime, a conformal transformation is defined as any transformation of the fields that changes the metric by no more than an overall factor. This is equivalent to the previous definitions (notice that it preserves light cones).

In general, conformal transformations may be grouped into:

- Translations,
- Rotations (also boosts, in spacetime),
- Dilatations (these just scale all lengths by a factor), and

- Special conformal transformations. (These are translations, but applied to an added point 'at infinity'. For instance, Imagine inverting all of 3-space through a unit sphere, so points stay on the same ray from the origin but points inside the sphere go outside and points outside come inside. A point a distance 2 from the origin goes to 1/2 and so on. A 'point at infinity' is added as the inverse of the origin. If you do an inversion, then a translation, then another inversion, it's a special conformal transformation because that sequence preserves angles).

Conformal gravity (or Weyl gravity) is a specific theory of gravity developed by Hermann Weyl in 1918. It is based on an action functional given by the integral of the square of the Weyl conformal curvature, that is, the part of the Riemann curvature tensor which is trace-free. Weyl showed that this part of the curvature is unchanged by a conformal transformation. Conformal gravity usually refers to this theory, but may refer to any gravity theory which is unchanged under conformal transformations.

Any solution to the vacuum Einstein equation is also a solution to Weyl gravity, but since the field equation of Weyl gravity (called the Bach equation [1]) is fourth order in derivatives of the metric, it also admits additional solutions.

Weyl gravity is not the only gravity theory that is invariant under conformal transformations. Ivanov and Niederle [2], found another formulation, and Wehner and Wheeler found another [3, 4].

1.2.2 The Weyl Tensor and Conformal Invariance

The Weyl tensor or conformal tensor in n dimensions, is defined ($n \geq 3$) by:

$$C_{abcd} = R_{abcd} + \frac{1}{n-2}(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{db} - g_{bd}R_{ca}) + \frac{1}{(n-2)(n-1)}(g_{ac}g_{db} - g_{ad}g_{cb})R.$$

Thus in four dimensions, this becomes;

$$C_{abcd} = R_{abcd} + \frac{1}{2}(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{db} - g_{bd}R_{ca}) + \frac{1}{6}(g_{ac}g_{db} - g_{ad}g_{cb})R.$$

The Weyl tensor possesses the same symmetries as the Riemann tensor, namely,

$$C_{abcd} = -C_{abdc} = -C_{bacd} = C_{cdab},$$

$$C_{abcd} + C_{adbc} + C_{acdb} = 0,$$

However, it possesses an additional symmetry, namely,

$$C^{a}{}_{bad} = 0$$

which states that the Weyl tensor is trace-free, in other words, it vanishes for any pair of contracted indices.

Two metrics g_{ab} and $\bar{g}_{ab}$ are said to be conformally related if,

$$g_{ab} = \Omega^2 \bar{g}_{ab}$$

where Ω(x) is a non-zero differentiable function. Given a manifold with two metrics defined on it which are conformal, the angles between vectors and the ratios of magnitudes of vectors, but not lengths, are the same for each metric. Moreover, the null geodesics of one metric coincide with the null geodesics of the other and the metrics also possess the same Weyl tensor.

$$C^{a}{}_{bcd} = \bar{\bar{C}}^{a}{}_{bcd}.$$

Any quantity which satisfies the relationship above is called conformally invariant. The metric, g_{ab}, the Christoffel symbol, Γ^{a}_{bc}, and the Riemann tensor, $R^{a}{}_{bcd}$ are examples of quantities which are not conformally invariant.

A metric is said to be conformally flat if it can be reduced to the form, $g_{ab} = \Omega^2 \eta_{ab}$, where η_{ab} is the flat metric.

1.2.3 The Weyl Action and Derivation of the Bach Equation

The field equation for Weyl gravity can be deduced by replacing the Ricci scalar in Einstein gravity by the square of the Weyl tensor as the Lagrangian. Hence we write the conformally invariant Weyl action as,

$$S = \alpha \int C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu} \sqrt{-g}\, d^4x.$$

We now wish to vary this action with respect to the metric. We start by rewriting the action as,

$$S = -\alpha \int C^{\alpha}_{\beta\mu\nu} C^{\beta}_{\alpha\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g}\, d^4x,$$

and then varying and requiring $\delta S = 0$ gives:

$$\delta S = -2\alpha \int \delta C^{\alpha}{}_{\beta\mu\nu} C^{\beta}{}_{\alpha\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} d^4x$$

$$- \alpha \int C^{\alpha}{}_{\beta\mu\nu} C^{\beta}{}_{\alpha\rho\sigma} \left(2g^{\mu\rho} \delta^{\nu}_{\lambda} \delta^{\sigma}_{\tau} - \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} g_{\lambda\tau} \right) \sqrt{-g} \delta g^{\lambda\tau} d^4x = 0 \quad (40)$$

The integrand of the second term becomes:

$$C^{\alpha}_{\ \beta\mu\nu} C^{\beta}{}_{\alpha\rho\sigma} \left(\delta^{\mu}_{\lambda} \delta^{\sigma}_{\tau} g^{\nu\sigma} - \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} g_{\lambda\tau} \right) = -2 \left(C^{\alpha\beta\mu}{}_{\lambda} C_{\alpha\beta\mu\tau} - \frac{1}{4} g_{\lambda\tau} C^{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu} \right) = 0$$

In the light of above, (40) becomes:

$$\delta S = -2\alpha \int \delta C^{\beta}{}_{\alpha\mu\nu} C^{\alpha}{}_{\beta\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} d^4x = 0 \quad (41)$$

The Weyl curvature $C^{\beta}{}_{\alpha\mu\nu}$ can be expressed in four dimensions in terms of the Riemann curvature as:

$$C^{\beta}{}_{\alpha\mu\nu} = R^{\beta}{}_{\alpha\mu\nu} - \frac{1}{2} \left(\delta^{\beta}_{\mu} R_{\alpha\nu} - g_{\alpha\mu} R^{\beta\nu} - \delta^{\beta}_{\upsilon} R_{\alpha\mu} + g_{\alpha\nu} R^{\beta\mu} \right) + \frac{1}{6} \left(g_{\alpha\nu} \delta^{\beta}_{\mu} - g_{\alpha\mu} \delta^{\beta}_{\upsilon} \right) R \quad (42)$$

Substituting (42) into (41) we get:

$$0 = -2\alpha \int \delta C^{\beta}{}_{\alpha\mu\nu} C^{\alpha}{}_{\beta\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} d^4x$$

$$= -2\alpha \int \delta \Big[R^{\beta}{}_{\alpha\mu\nu} + \frac{1}{2} \left(g_{\alpha\mu} R^{\beta}{}_{\upsilon} - g_{\alpha\nu} R^{\beta}{}_{\mu} - \delta^{\beta}_{\mu} R_{\alpha\nu} + \delta^{\beta}_{\upsilon} R_{\alpha\mu} \right)$$

$$- \frac{1}{6} \left(g_{\alpha\mu} \delta^{\beta}_{\upsilon} - g_{\alpha\upsilon} \delta^{\beta}_{\mu} \right) R \Big] C^{\alpha}{}_{\beta\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} d^4x.$$

$$= -2\alpha \int \Big[\delta R^{\beta}{}_{\alpha\mu\nu} + \frac{1}{2}\Big(g_{\alpha\mu}\delta R^{\beta}{}_{\upsilon} - g_{\alpha\nu}\delta R^{\beta}{}_{\mu} - \delta^{\beta}_{\mu}\delta R_{\alpha\nu} + \delta^{\beta}_{\upsilon}\delta R_{\alpha\mu}\Big)$$

$$- \frac{1}{6}\Big(g_{\alpha\mu}\delta^{\beta}_{\upsilon} - g_{\alpha\upsilon}\delta^{\beta}_{\mu}\Big)\delta R\Big] C^{\alpha}{}_{\beta}{}^{\mu\nu}\sqrt{-g}d^4x$$

$$+ 2\alpha \int \Big[-\frac{1}{2}\Big(\delta g_{\alpha\mu}R^{\beta}{}_{\upsilon} - \delta g_{\alpha\nu}R^{\beta}{}_{\mu}\Big) + \frac{1}{6}\Big(\delta g_{\alpha\mu}\delta^{\beta}_{\upsilon} - \delta g_{\alpha\upsilon}\delta^{\beta}_{\mu}\Big)R\Big] C^{\alpha}{}_{\beta}{}^{\mu\nu}\sqrt{-g}d^4x$$

$$= -2\alpha \int \Big[C^{\alpha}{}_{\beta}{}^{\mu\nu}\delta R^{\beta}{}_{\alpha\mu\nu} + \frac{1}{2}\Big(\delta g_{\alpha\mu}R^{\beta}{}_{\upsilon} - \delta g_{\alpha\nu}R^{\beta}{}_{\mu}\Big)C^{\alpha}{}_{\beta}{}^{\mu\nu}\Big]\sqrt{-g}d^4x \qquad (43)$$

We carry out the variation of the curvature in two steps. First, vary the connection:

$$\delta R^{\beta}{}_{\alpha\mu\nu} = D_{\nu}\delta\Gamma^{\beta}{}_{\alpha\mu} - D_{\mu}\delta\Gamma^{\beta}{}_{\alpha\nu}$$

Then substituting this into (43) and integrating by parts (discarding surface terms)

$$\delta S = -2\alpha \int \Big[C^{\alpha}{}_{\beta}{}^{\mu\nu}\Big(D_{\nu}\delta\Gamma^{\beta}{}_{\alpha\mu} - D_{\mu}\delta\Gamma^{\beta}{}_{\alpha\nu}\Big) + \frac{1}{2}\Big(\delta g_{\alpha\mu}R^{\beta}{}_{\upsilon} - \delta g_{\alpha\nu}R^{\beta}{}_{\mu}\Big)C^{\alpha}{}_{\beta}{}^{\mu\nu}\Big]\sqrt{-g}d^4x$$

$$= 2\alpha \int \Big[\Big(2D_{\nu}C^{\alpha}{}_{\beta}{}^{\mu\nu}\delta\Gamma^{\beta}{}_{\alpha\mu} - \delta g_{\alpha\mu}R^{\beta}{}_{\upsilon}C^{\alpha}{}_{\beta}{}^{\mu\nu}\Big)\Big]\sqrt{-g}d^4x \qquad (44)$$

The variation of the connection with respect to the metric is:

$$\delta\Gamma^{\alpha}{}_{\mu\upsilon} = \frac{1}{2}\delta g^{\alpha\beta}\big(g_{\beta\mu,\upsilon} + g_{\beta\upsilon,\mu} - g_{\mu\upsilon,\beta}\big) + \frac{1}{2}g^{\alpha\beta}\big(\delta g_{\beta\mu,\upsilon} + \delta g_{\beta\upsilon,\mu} - \delta g_{\mu\upsilon,\beta}\big)$$

$$= \delta g^{\alpha\beta}\Gamma_{\beta\mu\nu} + \frac{1}{2}g^{\alpha\beta}\left(\delta g_{\beta\mu;\upsilon} + \delta g_{\rho\mu}\Gamma^{\rho}{}_{\beta\nu} + \delta g_{\beta\rho}\Gamma^{\rho}{}_{\mu\nu}\right)$$

$$+ \frac{1}{2}g^{\alpha\beta}\left(\delta g_{\beta\upsilon;\mu} + \delta g_{\rho\nu}\Gamma^{\rho}{}_{\beta\mu} + \delta g_{\beta\rho}\Gamma^{\rho}{}_{\nu\mu}\right)$$

$$- \frac{1}{2}g^{\alpha\beta}\left(\delta g_{\mu\nu;\beta} + \delta g_{\rho\nu}\Gamma^{\rho}{}_{\mu\beta} + \delta g_{\mu\rho}\Gamma^{\rho}{}_{\nu\beta}\right)$$

$$= \frac{1}{2} g^{\alpha\beta}\left(\delta g_{\beta\mu;\upsilon} + \delta g_{\beta\upsilon;\mu} - \delta g_{\mu\nu;\beta}\right) + \delta g^{\alpha\beta}\Gamma_{\beta\mu\nu} + \frac{1}{2} g^{\alpha\beta}\delta g_{\beta\rho}\Gamma^{\rho}{}_{\nu\mu} + \frac{1}{2} g^{\alpha\beta}\delta g_{\beta\rho}\Gamma^{\rho}{}_{\nu\mu}$$

$$= \frac{1}{2} g^{\alpha\beta}\left(D_\upsilon \delta g_{\beta\mu} + D_\mu \delta g_{\beta\nu} - D_\beta \delta g_{\mu\nu}\right) + \delta g^{\alpha\beta}\Gamma_{\beta\mu\nu} - \frac{1}{2}\delta g^{\alpha\beta}\Gamma_{\beta\mu\nu} - \frac{1}{2}\delta g^{\alpha\beta}\Gamma_{\beta\nu\mu}$$

$$= \frac{1}{2} g^{\alpha\beta}\left(D_\upsilon \delta g_{\beta\mu} + D_\mu \delta g_{\beta\nu} - D_\beta \delta g_{\mu\nu}\right) \tag{45}$$

Substituting (45) into (44), we obtain:

$$0 = 2\alpha \int \left[2D_\nu C^{\alpha}{}_{\beta}{}^{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta}\left(D_\mu \delta g_{\rho\alpha} + D_\alpha \delta g_{\rho\mu} - D_\rho \delta g_{\alpha\mu}\right)\right) - \delta g_{\alpha\mu} R^{\beta}{}_{\upsilon} C^{\alpha}{}_{\beta}{}^{\mu\nu} \right] \sqrt{-g}\, d^4x$$

$$= 2\alpha \int \left[2D_\nu C^{\alpha\rho\mu\nu}\left(D_\mu \delta g_{\rho\alpha} + D_\alpha \delta g_{\rho\mu} - D_\rho \delta g_{\alpha\mu}\right) - \delta g_{\alpha\mu} R^{\beta}{}_{\upsilon} C^{\alpha}{}_{\beta}{}^{\mu\nu} \right] \sqrt{-g}\, d^4x$$

$$= 2\alpha \int \Big[-D_\mu D_\nu C^{\alpha\rho\mu\nu}\delta g_{\rho\alpha} - D_\alpha D_\nu C^{\alpha\rho\mu\nu}\delta g_{\rho\mu} + D_\rho D_\nu C^{\alpha\rho\mu\nu}\delta g_{\alpha\mu}$$

$$- \delta g_{\alpha\mu} R^{\beta}{}_{\upsilon} C^{\alpha}{}_{\beta}{}^{\mu\nu} \Big] \sqrt{-g}\, d^4x$$

$$= 2\alpha \int \left[D_\alpha D_\nu C^{\rho\alpha\mu\nu}\delta g_{\rho\mu} + D_\rho D_\nu C^{\alpha\rho\mu\nu}\delta g_{\alpha\mu} - \delta g_{\alpha\mu} R^{\beta}{}_{\upsilon} C^{\alpha}{}_{\beta}{}^{\mu\nu} \right] \sqrt{-g}\, d^4x$$

$$= 2\alpha \int \left[2D_\rho D_\nu C^{\alpha\rho\mu\nu} - R^{\beta}{}_{\upsilon} C^{\alpha}{}_{\beta}{}^{\mu\nu} \right] \delta g_{\alpha\mu} \sqrt{-g}\, d^4x$$

This gives the Bach equation

$$R_{\mu\nu} C^{\alpha\mu\beta\nu} - 2D_\mu D_\nu C^{\alpha\mu\beta\nu} = 0$$

$$W^{\mu\nu} = R_{\kappa\lambda} C^{\kappa\mu\lambda\nu} - 2\nabla_\kappa \nabla_\lambda C^{\kappa\mu\lambda\nu} \tag{46}$$

where $\nabla_\mu \nabla_\nu$ is another notation for $D_\mu D_\nu$.

The gravitational field equations in the presence of matter are then expected to take the form

$$W_{\mu\nu} = \frac{\alpha}{2} T_{\mu\nu}. \tag{47}$$

1.3 EINSTEIN GRAVITY VERSUS WEYL GRAVITY

A spacetime is called flat if its Riemann curvature tensor vanishes i.e. $R_{abcd} = 0$, and is called conformally flat if there exists a conformal transformation such that $\tilde{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$ such that the Riemann curvature tensor in the space with metric $\tilde{g}_{\alpha\beta}$ vanishes i.e. $\tilde{R}_{abcd} = 0$.

A necessary and sufficient condition for a metric to be conformally flat is that its Weyl tensor vanishes everywhere. Any two-dimensional Riemannian manifold is conformally flat [5]

An *n*-dimensional space is said to be an Einstein space if the trace-free part of the Ricci tensor is identically zero i.e. i.e. $R_{ab} - \frac{1}{n} g_{ab} R = 0$. Similarly, an *n*-dimensional space with metric g_{ab} is said to be a conformal Einstein space (or conformally Einstein) if there exists a conformal transformation $\tilde{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$ such that in the conformal space $\tilde{R}_{ab} - \frac{1}{n} \tilde{g}_{ab} \tilde{R} = 0$ [6].

As stated earlier, general relativity and Weyl gravity are two different ways of explaining gravity. They do have certain similarities and differences.

Weyl tensor encodes the stretching and squeezing property of spacetime curvature. Since the Einstein equation shows that the trace parts of the Riemann curvature are given by the energy-momentum tensor of matter fields, the Weyl curvature represents the purely gravitational field.

Gravitational waves have a property of squeezing and stretching and appear in Einstein's solutions in free space. Since in this state, the Ricci part of the Riemann tensor vanishes, the Weyl part is responsible for the gravitational waves.

It is possible to rewrite the Weyl action in the form;

$$S = \alpha \int \left(3R_{\alpha\beta}R^{\alpha\beta} - R^2\right)\sqrt{-g}\,d^4x.$$

This form leads to field equations depending only on the Ricci tensor, so that vacuum solutions to general relativity are also solutions to Weyl gravity. Examples show that the converse does not hold. A good example is the Friedmann-Walker-Roberts metric which solves the vacuum Bach equation but represents a perfect fluid in general relativity.

Conformal gravity is a four-derivative theory which makes the quantized version of the theory more convergent and renormalizable even though it may result to issues related to causality.

If we consider two metrics,

$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$ and $ds^2 = e^{2\phi}(-c^2dt^2 + dx^2 + dy^2 + dz^2)$,

the first metric is flat while the Riemann curvature and Ricci tensor do not vanish for the second in general. In general relativity, two metrics which are related by such a factor

$$\tilde{g}_{\alpha\beta} = e^{2\phi}g_{\alpha\beta}.$$

are treated as different solutions. In Weyl gravity, if the Bach tensor vanishes for $g_{\alpha\beta}$ it will also vanish for $\tilde{g}_{\alpha\beta}$. These two metrics are treated as equivalent physically.

1.4 BIRKOFF'S THEOREM

H.-J. Schmidt has classified Birkhoff-type theorems into four classes [8] as follows: "In field theory, it depicts the absence of helicity 0- and spin 0- parts of the gravitational field. In relativistic astrophysics, it states that the gravitational far-field of a spherically symmetric star apart from its mass, carries no information about the star; therefore, a radially oscillating star has a static gravitational far-field. In differential geometry, it comes in the form: every member of a family of pseudo-Riemannian spacetimes has more isometries than expected from the original metric ansatz. Finally, in mathematical physics, Birkhoff's theorem states: up to singular exceptions of measure zero, the spherically symmetric solutions of Einstein's vacuum field equation with $\Lambda = 0$ can be expressed by the Schwarzschild metric; for $\Lambda \neq 0$, it is the Schwarzschild-de Sitter metric instead."

In other words, the fourth statement states that the exterior solution depicting the spacetime outside of a spherically, nonrotating, gravitating body must be given by the Schwarzschild metric.

1.5 RESEARCH AIMS AND OBJECTIVES

The aim of this study is to reproduce Birkhoff's theorem for Weyl gravity. We use the spherically-symmetric vacuum equations of conformal gravity. In particular, our main objective is to focus on:

a. Understanding the consequences of Birkhoff's theorem for both general relativity and Weyl gravity
b. Using new computer tools (Maple and Mathematica) [9, 10], to explore the properties of the Birkhoff theorem in Weyl gravity, and
c. Discussing the physical implications of the new solutions obtained such as killing vectors, geodesics, and effective potential.

1.6 SCOPE AND METHODOLOGY

We have adopted the analytical and algebraic methods in solving our problems and used the Maple and Mathematica programs [9, 10], to verify our results. We begin by explicitly showing how Birkhoff's theorem holds for general relativity and then examined the properties of the corresponding solution in conformal gravity.

CHAPTER TWO

LITERATURE REVIEW

2.1 FROM EINSTEIN TO WEYL

Gravity is known today to be the oldest force. Although weak, yet it is long ranged and couples to everything as long as it has mass or energy. Gravity cannot be screened since its potential is always attractive and is found to be mediated by a particle with zero mass; *the graviton.* Einstein's gravity and conformal gravity have both been successful in explaining general relativity, which accounts in exquisite detail for all gravitational phenomena [11]. Also, solutions in general relativity are also solutions in conformal gravity. Therefore, the conformal gravity (CG), derived from the action quadratic in the Weyl tensor, may well be a viable alternative to Einstein's gravity as the fundamental theory of gravitation [12]. The underlying theory behind CG, may supply the proper framework for a solution f some of the most challenging problems of theoretical physics such as cosmological constant, the dark matter, and the dark energy [14].

After a period of development, from the introduction of the equivalence principle in 1908 through the presentation of the field equations in 1915, tremendous progress was observed in the field of general relativity [14].

Weyl, in 1918 [15] and Bach, in 1921 [1] developed a theory of gravity based on the conformally invariant Weyl action, which when varied with the principle of least action leads to the Bach equation. Bach also used an alternate approach [1] to write the equivalent action and field equations, depending only the Ricci tensor and its derivatives by using the Gauss-Bonnet integral for the Euler character [16]

In 1921, Bach found the spherically symmetric solutions of conformal gravity and showed that the solution includes the familiar exterior Schwarzschild solution as a special case and contains an extra gravitational potential term which grows linearly with distance [1]. More recently, Riegert showed that conformal gravity admits a Birkhoff theorem [17]. Other CG studies yielded methods for determining the structure of the gravitational equations [18].

2.2 KNOWN SOLUTIONS

Vacuum solutions of the Bach equation have been studied extensively. It has been claimed [19] that every spherically symmetric solution of the Bach equation is almost everywhere conformally related to an Einstein space. Every spherically symmetric solution of the Bach equation is almost everywhere trivial [20]. New vacuum solutions have been discovered using the 2+2 decomposition of the Bach equation [19].

More recent studies in CG investigate the cylindrically symmetric [13] and the exact String-like solutions [21] in conformal gravity which could possibly be used to analyze the light bending in the vicinity of localized linear sources in CG [22]. The analogous problem of light bending in spherically-symmetric gravitational field was only settled recently [21]

CHAPTER THREE

BIRKHOFF'S THEOREM ANALYSIS

3.1 THE SCHWARSZCHILD SOLUTION

Most often, the Schwarzschild solution is the exterior solution for a normal star or planet. If we want to know what general relativity says about Earth's gravity, we would do two solutions - one for the interior of the planet, with the matter given by some sort of mass density, and a second for the exterior region where $T^{ab} = 0$. Since Earth is spherical, the exterior solution will be Schwarzschild. That solution would then be matched to the matter solution for the interior.

Even if we don't have that interior solution, we can tell which Schwarzschild solution is right by looking at the Newtonian limit of an orbit far from the planet. That's how we usually find a mass in the solutions as M and also the universal gravitational constant appears as will be discussed.

The black hole is a special case where the density becomes so high that light can't escape. The original studies of black holes begin with a large star that collapses, so again we have interior and exterior solutions. That calculation is exactly parallel to Chandrasekhar's 1930 proof that a white dwarf star will collapse to form a neutron star. A similar calculation shows that a sufficiently massive neutron star is unstable against collapse that will carry the matter within the horizon. If that happens, there is no known force that can keep the matter from collapsing completely to the center. Surely, at that point something else happens that we know nothing about - perhaps the matter at the center enters a string-like state. We don't know because experiments haven't probed densities that great. However, we do know that down to an extremely small size the laws of particle interaction that we do know show us that the collapse continues well within the Schwarzschild horizon i.e. where the black holes are found.

Far from a black hole, the gravitation is just like it would be for a large normal star of the same mass. This is really what determines M for the black hole, even though we don't know what lies at the center.

We begin with the general spherically symmetric static line-element of the form

$$ds^2 = -f^2(r)dt^2 + g^2(r)dr^2 + r^2(r)d\theta + r^2 sin^2\theta(r)d\varphi^2 \tag{48}$$

$$g_{ab} = \begin{bmatrix} -f^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{bmatrix} \tag{49}$$

$$g^{a\,b} = \begin{bmatrix} -f^{-2} & 0 & 0 & 0 \\ 0 & g^{-2} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2} cosec^2\theta \end{bmatrix} \tag{50}$$

We aim at finding the Schwarzschild Solutions from the basics by following the steps shown in appendix I and the results are summarized below:

Step 1:

We first we find the non-zero components of the Affine connection or Christoffel symbol $\Gamma_{\alpha\mu\nu}$, using (9). The results are summarized in the table below with $\Gamma_{001} = -ff'$, and the derivative of f with respect to r is given by f' and so on.

S/N	$\Gamma_{\alpha\mu\nu}$	Corresponding Value
1	Γ_{001}	$-ff'$
2	Γ_{010}	$-ff'$
3	Γ_{100}	ff'
4	Γ_{111}	gg'
5	Γ_{221}	r
6	Γ_{212}	r
7	Γ_{122}	$-r$
8	Γ_{331}	$rsin^2\theta$
9	Γ_{313}	$rsin^2\theta$
10	Γ_{133}	$-rsin^2\theta$
11	Γ_{332}	$r^2sin\theta cos\theta$
12	Γ_{233}	$-r^2sin\theta cos\theta$
13	Γ_{323}	$r^2sin\theta cos\theta$

Table 3.1: non-zero components for the affine connection

Step 2:

Using the results in table 3.1 above, we can also compute the corresponding mixed form of the affine connection as summarized in table 3.2 below

S/N	$\Gamma^{\alpha}{}_{\mu\nu} = g^{\alpha\beta}\Gamma_{\beta\mu\nu}$	Corresponding Value
1	$\Gamma^{0}{}_{01}$	$\frac{f'}{f}$
2	$\Gamma^{0}{}_{10}$	$\frac{f'}{f}$
3	$\Gamma^{1}{}_{00}$	$\frac{ff'}{g^2}$
4	$\Gamma^{1}{}_{11}$	$\frac{g'}{g}$
5	$\Gamma^{2}{}_{21}$	$\frac{1}{r}$
6	$\Gamma^{2}{}_{12}$	$\frac{1}{r}$
7	$\Gamma^{1}{}_{22}$	$-\frac{r}{g^2}$
8	$\Gamma^{3}{}_{31}$	$\frac{1}{r}$
9	$\Gamma^{3}{}_{13}$	$\frac{1}{r}$
10	$\Gamma^{1}{}_{33}$	$-\frac{rsin^2\theta}{g^2}$
11	$\Gamma^{3}{}_{32}$	$cot\theta$
12	$\Gamma^{2}{}_{33}$	$-sin\theta cos\theta$
13	$\Gamma^{3}{}_{23}$	$cot\theta$

Table 3.2: non-zero components for the mixed affine connection

Step 3:

We compute the curvature and the results are summarized in the table below:

S/N	$R^{\alpha}{}_{\beta\mu\nu}$	Corresponding Value
1	$R^{0}{}_{101}$	$\frac{g'ff' - gff''}{gf^2}$
2	$R^{1}{}_{010}$	$\frac{gff'' - g'ff'}{g^3}$
3	$R^{0}{}_{202}$	$-\left(\frac{f'r}{fg^2}\right)$
4	$R^{2}{}_{020}$	$\frac{ff'}{rg^2}$
5	$R^{0}{}_{303}$	$-\frac{f'rsin^2\theta}{fg^2}$
6	$R^{3}{}_{030}$	$\frac{ff'}{rg^2}$
7	$R^{1}{}_{212}$	$\frac{rg'}{g^3}$
8	$R^{2}{}_{121}$	$\frac{g'}{rg}$
9	$R^{1}{}_{313}$	$\frac{rg'sin^2\theta}{g^3}$
10	$R^{3}{}_{131}$	$\frac{g'}{rg}$
11	$R^{2}{}_{323}$	$sin^2\theta\left(1-\frac{1}{g^2}\right)$

12	${R^3}_{232}$	$1-\left(\frac{1}{g^2}\right)$

Table 3.3: non-zero components for the Riemann Curvature Tensor.

Step 4:

The Ricci tensor components for the metric (49) are found and listed in table 3.4 below:

S/N	$R_{\mu\nu} = {R^\alpha}_{\mu\alpha\nu}$	Corresponding Value
1	R_{00}	$\frac{rgff'' - rg'ff' + 2gff'}{rg^3}$
2	R_{11}	$\frac{rg'ff' - rgff'' + 2f^2g'}{rgf^2}$
3	R_{22}	$\frac{-grf' + rg'f + fg^3 - fg}{fg^3}$
4	R_{33}	$\frac{-gf'rsin^2\theta + rg'fsin^2\theta + fg^3sin^2\theta - fgsin^2\theta}{fg^3}$

Table 3.4: non-zero components of the Ricci tensor.

Step 5:

We now solve the Einstein equation. For vacuum solutions, the scalar curvature must vanish and the Einstein equation is equivalent vanishing Ricci tensor. Equating each nonvanishing component in table 3,4 above to zero, and cancelling overall common factors, we get:

$$ds^2 = -\left(\frac{1}{c}\sqrt{1-\frac{1}{rc'}}\right)^2 dt^2 + \left(\frac{1}{\sqrt{1-\frac{1}{rc'}}}\right)^2 dr^2 + r^2 d\Omega^2$$

$$= -\frac{1}{c^2}\left(1-\frac{1}{rc'}\right)dt^2 + \left(\frac{1}{1-\frac{1}{rc'}}\right)dr^2 + r^2 d\Omega^2 \qquad (51)$$

Step 6:

Next, we determine the remaining constant, c'. To do this we require the geodesic equation, which follows by finding the extremum of the arclength. This gives the Schwarzschild solution

$$\Rightarrow c' = \frac{1}{2GM}$$

$$\Rightarrow f = \sqrt{1-\frac{1}{rc'}} = \sqrt{1-\frac{2GM}{r}}$$

$$\Rightarrow ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2GM}{r}\right)} + r^2 d\Omega^2 \qquad (52)$$

Full details of hidden details for all steps 1 to 6 are given in appendix I.

3.2 BIRKHOFF'S THEOREM IN GENERAL RELATIVITY

We have chosen the general time-dependent spherically symmetric line-element of the form

$$ds^2 = -f^2(r,t)dt^2 + g^2(r,t)dr^2 + r^2(r)d\theta + r^2 sin^2\theta(r)d\varphi^2 \qquad (53)$$

$$g_{ab} = \begin{bmatrix} -f^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{bmatrix} \qquad (54)$$

We aim at finding the Schwarzschild Solutions from the basics by following the steps below:

Step 1:

We first we find the non-zero components of the Affine connection or Christoffel symbol.

The results are summarized in the table below:

S/N	$\Gamma_{\alpha\mu\nu}$	Corresponding Value
1	Γ_{000}	$-f\dot{f}$
2	Γ_{001}	$-ff'$
3	Γ_{010}	$-ff'$
4	Γ_{100}	ff'
5	Γ_{110}	$g\dot{g}$
6	Γ_{101}	$g\dot{g}$
7	Γ_{011}	$-g\dot{g}$
8	Γ_{111}	gg'

9	Γ_{221}	r
10	Γ_{212}	r
11	Γ_{122}	$-r$
12	Γ_{331}	$rsin^2\theta$
13	Γ_{313}	$rsin^2\theta$
14	Γ_{133}	$-rsin^2\theta$
15	Γ_{332}	$r^2sin\theta cos\theta$
16	Γ_{233}	$-r^2sin\theta cos\theta$
17	Γ_{323}	$r^2sin\theta cos\theta$

Table3.5: Non zero Components of the Affine Connection.

Step 2:

We compute the corresponding mixed form of the Affine connection. The results are summarized in the table below:

S/N	$\Gamma^{\alpha}{}_{\mu\nu} = g^{\alpha\beta}\Gamma_{\beta\mu\nu}$	Corresponding Value
1	$\Gamma^{0}{}_{00}$	$\frac{\dot{f}}{f}$
2	$\Gamma^{0}{}_{01}$	$\frac{f'}{f}$
3	$\Gamma^{0}{}_{10}$	$\frac{f'}{f}$

4	$\Gamma^{1}{}_{00}$	$\frac{ff'}{g^2}$
5	$\Gamma^{1}{}_{10}$	$\frac{\dot{g}}{g}$
6	$\Gamma^{1}{}_{01}$	$\frac{\dot{g}}{g}$
7	$\Gamma^{0}{}_{11}$	$\frac{g\dot{g}}{f^2}$
8	$\Gamma^{1}{}_{11}$	$\frac{g'}{g}$
9	$\Gamma^{2}{}_{21}$	$\frac{1}{r}$
10	$\Gamma^{2}{}_{12}$	$\frac{1}{r}$
11	$\Gamma^{1}{}_{22}$	$-\frac{r}{g^2}$
12	$\Gamma^{3}{}_{31}$	$\frac{1}{r}$
13	$\Gamma^{3}{}_{13}$	$\frac{1}{r}$
14	$\Gamma^{1}{}_{33}$	$-\frac{rsin^2\theta}{g^2}$
15	$\Gamma^{3}{}_{32}$	$cot\theta$
16	$\Gamma^{2}{}_{33}$	$-sin\theta cos\theta$
17	$\Gamma^{3}{}_{23}$	$cot\theta$

Table3.6: Non zero Components of the Mixed Affine Connection.

Step 3:

We proceed to compute the curvature. The results are summarized in the table below:

S/N	$R^{\alpha}{}_{\beta\mu\nu}$	Corresponding Value
1	$R^{0}{}_{101}$	$\frac{g^2 f\ddot{g} - g^2\dot{g}\dot{f} - gf^2 f'' + g'f^2 f'}{gf^3}$
2	$R^{1}{}_{010}$	$\frac{gf^2 f'' - g'f^2 f' - g^2 f\ddot{g} + g^2\dot{g}\dot{f}}{g^3 f}$
3	$R^{0}{}_{202}$	$-\left(\frac{f'r}{fg^2}\right)$
4	$R^{2}{}_{020}$	$\frac{ff'}{rg^2}$
5	$R^{0}{}_{303}$	$-\frac{f'rsin^2\theta}{fg^2}$
6	$R^{3}{}_{030}$	$\frac{ff'}{rg^2}$
7	$R^{1}{}_{212}$	$\frac{rg'}{g^3}$
8	$R^{2}{}_{121}$	$\frac{g'}{rg}$
9	$R^{1}{}_{313}$	$\frac{rg'sin^2\theta}{g^3}$
10	$R^{3}{}_{131}$	$\frac{g'}{rg}$
11	$R^{2}{}_{323}$	$sin^2\theta\left(1 - \frac{1}{g^2}\right)$

12	${R^3}_{232}$	$1-\left(\frac{1}{g^2}\right)$

Table3.7: Non zero Components of the Riemann Curvature Tensor

Step 4:

We proceed to compute the Ricci tensors and the results are summarized in the table below:

S/N	$R_{\mu\nu} = {R^{\alpha}}_{\mu\alpha\nu}$	Corresponding Value
1	R_{00}	$\frac{rgf^2f'' - rg'f^2f' - rg^2f\ddot{g} + rg^2\dot{g}\dot{f} + 2gf^2f'}{g^3fr}$
2	R_{11}	$\frac{rg^2f\ddot{g} - rg^2\dot{g}\dot{f} - rgf^2f'' + rg'f^2f' + 2f^3g'}{rgf^3}$
3	R_{22}	$\frac{-grf' + rg'f + fg^3 - fg}{fg^3}$
4	R_{33}	$\frac{-gf'rsin^2\theta + rg'fsin^2\theta + fg^3sin^2\theta - fgsin^2\theta}{fg^3}$

Table3.8: Non zero Components of the Ricci Tensor

Step 5:

Solving the differential equations we get;

$$ds^2 = -\left(k(t)\sqrt{1-\frac{2GM}{r}}\right)^2 dt^2 + \left(\frac{1}{\sqrt{1-\frac{2GM}{r}}}\right)^2 dr^2 + r^2 d\Omega^2$$

$$= -\left(\sqrt{1-\frac{2GM}{r}}\right)^2 k^2(t)dt^2 + \left(\frac{1}{\sqrt{1-\frac{2GM}{r}}}\right)^2 dr^2 + r^2 d\Omega^2$$

$$\Rightarrow ds^2 = -\left(1-\frac{2GM}{r}\right)k^2(t)dt^2 + \frac{dr^2}{\left(1-\frac{2GM}{r}\right)} + r^2 d\Omega^2 \tag{54}$$

$let\ dt' = k(t)dt$

$$\Rightarrow ds^2 = -\left(1-\frac{2GM}{r}\right)dt'^2 + \frac{dr^2}{\left(1-\frac{2GM}{r}\right)} + r^2 d\Omega^2 \tag{55}$$

The equality of (55) and (52) proves the Birkhoff theorem for general relativity.

3.3 BIRKHOFF'S THEOREM IN WEYL GRAVITY

Several attempts have been made to prove that Birkhoff's theorem also holds in Weyl gravity. The first attempt was by Riegert and Ronald in [15]. In this paper, it was derived that the most general spherically symmetric, electrovac solution is static.

3.3.1 Conformal Transformation of the Static Spherically Symmetric Metric

We can find a conformal transformation to simplify the usual form of the Schwarzschild ansatz,

$$ds^2 = -f^2(r)dt^2 + g^2(r)dr^2 + r^2(r)d\theta + r^2 sin^2\theta(r)d\varphi^2 \quad (56)$$

$$g_{ab} = \begin{bmatrix} -f^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{bmatrix} \quad (57)$$

$$g^{a\,b} = \begin{bmatrix} -f^{-2} & 0 & 0 & 0 \\ 0 & g^{-2} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2} cosec^2\theta \end{bmatrix} \quad (58)$$

Or simply,

$$ds^2 = -f^2(r)dt^2 + g^2(r)dr^2 + r^2(r)d\Omega^2 \quad (59)$$

Where,

$$r^2 d\Omega^2 = r^2(r)d\theta + r^2 sin^2\theta(r)d\varphi^2$$

Perform a conformal transformation by h^2,

$$h^2 ds^2 = -f^2(r)dt^2 + g^2(r)dr^2 + r^2(r)d\Omega^2$$

$$d\tilde{s}^2 = -h^2 f^2 dt^2 + h^2 g^2 dr^2 + h^2 r^2 d\Omega^2$$

$$= -h^2 f^2 dt^2 + h^2 g^2 dr^2 + R^2 d\Omega^2$$

Where, $R^2 = h^2 r^2$

$$d\tilde{s}^2 = h^2 ds^2$$

where we redefine the radial coordinate $R^2 = h^2 r^2$. Then,

$$r = \frac{R}{h(R)}$$

$$dr = \left(\frac{1}{h} - \frac{h'}{h^2}\right) dR$$

Now, regarding f, g, and h as functions of R, we have:

$$d\tilde{s}^2 = -h^2 f^2 dt^2 + g^2 \left(1 - \frac{h'}{h}\right)^2 dR^2 + R^2 d\Omega^2$$

We would like to get this reduced to a convenient form. Require,

$$g^2 \left(1 - \frac{h'}{h}\right)^2 = \frac{1}{h^2 f^2}$$

$$1 - \frac{h'}{h} = \frac{1}{hgf}$$

$$h' - h + \frac{1}{gf} = 0$$

To solve this, let $h = e^{\phi}$. Then,

$$h = e^{\phi}$$

$$h' = \phi' e^{\phi}$$

so that,

$$h' - h + \frac{1}{gf} = 0$$

$$\phi' e^{\phi} - e^{\phi} + \frac{1}{gf} = 0$$

$$\phi' - 1 + \frac{1}{gf} e^{-\phi} = 0$$

Now let $\psi = \phi - R$ so that $\psi' = \phi' - 1$. The equation now becomes:

$$\psi' + \frac{1}{gf} e^{-(\psi + R)} = 0$$

$$e^{\psi} \psi' + \frac{1}{gf} e^{-R} = 0$$

Finally, let $H = e^{\psi}$,

$$H' + \frac{1}{gf} e^{-R} = 0$$

$$H' = -\frac{1}{gf} e^{-R}$$

$$H = -\int \frac{1}{gf} e^{-R} \, dR$$

Now undo the transformations to find h.

$$e^{\psi} = -\int \frac{1}{gf} e^{-R} \, dR$$

$$e^{(\phi - R)} = -\int \frac{1}{gf} e^{-R} \, dR$$

$$e^{\phi} = -e^{R} \int \frac{1}{gf} e^{-R} \, dR$$

$$h = -e^{R} \int \frac{1}{gf} e^{-R}\, dR$$

This shows that the required h exists.

Next we return to the line element,

$$d\tilde{s}^2 = -h^2 f^2 dt^2 + g^2 \left(1 - \frac{h'}{h}\right)^2 dR^2 + R^2 d\Omega^2$$

Where $g^2 \left(1 - \frac{h'}{h}\right)^2 = \frac{1}{h^2 f^2}$. This eliminates g,

$$d\tilde{s}^2 = -h^2 f^2 dt^2 + g^2 \left(1 - \frac{h'}{h}\right)^2 dR^2 + R^2 d\Omega^2$$

$$d\tilde{s}^2 = -h^2 f^2 dt^2 + \frac{1}{h^2 f^2} dR^2 + R^2 d\Omega^2$$

Finally, we define a single new function

$$F(R) \equiv h^2 f^2$$

to put the line element in the form

$$ds^2 = -F(R) dt^2 + \frac{1}{F(R)} dR^2 + R^2 d\Omega^2 \tag{60}$$

We now proceed to solve the Bach equation in this form.

3.3.2 The Bach Equations for metric in (60)

Maple gives one of the Bach equations to be

$$-\frac{1}{24fr^4}\left\{2\left(\frac{d^3f}{dr^3}\right)\left(\frac{df}{dr}\right)r^4-\left(\frac{d^2f}{dr^2}\right)^2 r^4-4\left(\frac{d^3f}{dr^3}\right)fr^3+4\left(\frac{d^2f}{dr^2}\right)\left(\frac{df}{dr}\right)r^3\right\}$$

$$-\frac{1}{24fr^4}\left\{-4\left(\frac{d^2f}{dr^2}\right)^2 fr^2-4\left(\frac{df}{dr}\right)^2 r^2+8f\left(\frac{df}{dr}\right)r-4f^2+4\right\}$$

and for this equation give a solution

$$f=c_3r^2+c_1+\frac{c_1}{r}+\frac{(c_1-1)(c_1+1)}{3c_2r^2} \tag{61}$$

3.3.3 Conformal Transformation of the Time-Dependent Spherically Symmetric Metric

We can find a conformal transformation to simplify the usual form of spherically symmetric ansatz,

$$ds^2=-f^2(r,t)dt^2+g^2(r,t)dr^2+r^2(r,t)d\Omega^2$$

by performing a conformal transformation by $\frac{1}{h^2(R,T)}$

$$d\tilde{s}^2=\frac{1}{h^2}ds^2$$

$$=-\frac{1}{h^2}f^2dt^2+\frac{1}{h^2}g^2dr^2+\frac{1}{h^2}r^2d\Omega^2$$

$$=-\frac{1}{h^2}f^2dt^2+\frac{1}{h^2}g^2dr^2+R^2d\Omega^2$$

where $R^2 = \frac{r^2}{h^2}$.

Then, let

$r = h(R,T)R$

$dr = \dot{h}dT + (h' + h)dR$

$t = h(R,T)T$

$dt = h'dR + \left(\dot{h} + h\right)dT$

Then,

$$d\tilde{s}^2 = -\frac{1}{h^2}f^2\left(h'dR + \left(\dot{h} + h\right)dT\right)^2 + \frac{1}{h^2}g^2\left(\dot{h}dT + (h' + h)dR\right)^2 + R^2 d\Omega^2$$

$$= -\frac{1}{h^2}f^2\left({h'}^2 dR^2 + 2h'\left(\dot{h} + h\right)dRdT + \left(\dot{h} + h\right)^2 dT^2\right)$$

$$+\frac{1}{h^2}g^2\left(\dot{h^2}dT^2 + 2\dot{h}(h' + h)dRdT + (h' + h)^2 dR^2\right) + R^2 d\Omega^2$$

$$= -\left(\frac{1}{h^2}f^2\left(\dot{h} + h\right)^2 - \frac{1}{h^2}g^2\dot{h^2}\right)dT^2 + \left(\frac{1}{h^2}g^2 2\dot{h}(h' + h) - \frac{1}{h^2}f^2 2h'\left(\dot{h} + h\right)\right)dRdT$$

$$+\left(\frac{1}{h^2}g^2(h' + h)^2 - \frac{1}{h^2}f^2 h'^2\right)dR^2 + R^2 d\Omega^2$$

We would like to solve for h so that,

$$\frac{1}{h^2}g^2 2\dot{h}(h' + h) = \frac{1}{h^2}f^2 2h'\left(\dot{h} + h\right)$$

$$\frac{1}{h^2}g^2(h' + h)^2 - \frac{1}{h^2}f^2 h'^2 = \frac{1}{\frac{1}{h^2}f^2\left(\dot{h} + h\right)^2 - \frac{1}{h^2}g^2\dot{h^2}}$$

These two choices both diagonalize the metric, and equate the coefficient of dT^2 with the inverse of the coefficient of dR^2.

Let $h = e^{\phi}$,

$g^2(\phi' + 1)\dot{\phi} = f^2\phi'(\dot{\phi} + 1)$

$$1 = \left(g^2(\phi' + 1)^2 - f^2\phi'^2\right)\left(f^2\left(\dot{\phi} + 1\right)^2 - g^2\dot{\phi}^2\right)$$

We now solve for $\dot{\phi}$:

$$(g^2(\phi' + 1) - f^2\phi')\dot{\phi} = f^2\phi'$$

$$\dot{\phi} = \frac{f^2\phi'}{g^2(\phi' + 1) - f^2\phi'}$$

Then the second equation becomes:

$$1 = \left(g^2(\phi' + 1)^2 - f^2\phi'^2\right)\left(f^2\left(\frac{f^2\phi'}{g^2(\phi' + 1) - f^2\phi'} + 1\right)^2 - g^2\left(\frac{f^2\phi'}{g^2(\phi' + 1) - f^2\phi'}\right)^2\right)$$

$$(g^2(\phi' + 1) - f^2\phi')^2 = \left(g^2(\phi' + 1)^2 - f^2\phi'^2\right)\left(f^2g^4(\phi' + 1)^2 - g^2f^4\phi'^2\right)$$

$$g^4(\phi' + 1)^2 - 2g^2f^2\phi'(\phi' + 1) + f^2f^2\phi'^2$$
$$= f^2g^6(\phi' + 1)^4 - 2f^4g^4\phi'^2(\phi' + 1)^2 + f^6g^2\phi'^4$$

$$g^4(\phi' + 1)^2 - 2g^2f^2\phi'(\phi' + 1) + f^4\phi'^2$$
$$= f^2g^2\left(g^4(\phi' + 1)^4 - 2f^2g^2\phi'^2(\phi' + 1)^2 + f^4\phi'^4\right)$$

$$(g^2(\phi' + 1) - f^2\phi') = fg\left(g^2(\phi' + 1)^2 - f^2\phi'^2\right)$$

Therefore, solving for ϕ',

$$g^2\phi' + g^2 - f^2\phi' = fg^3\phi'^2 + 2fg^3\phi' + fg^3 - f^3g\phi'^2$$

$$0 = fg(g^2 - f^2)\phi'^2 + (2fg^3 + f^2 - g^2)\phi' + (fg - 1)g^2$$

$$\phi' = \frac{1}{2fg(g^2 - f^2)}\left(-(2fg^3 + f^2 - g^2) \pm \sqrt{(2fg^3 + f^2 - g^2)^2 - 4fg(g^2 - f^2)(fg - 1)}\right)$$

$$\phi = \int \frac{dR}{2fg(g^2 - f^2)}\Big(-(2fg^3 + f^2 - g^2)$$
$$\pm \sqrt{(2fg^3 + f^2 - g^2)^2 - 4fg(g^2 - f^2)(fg - 1)}\Big)$$

Integrating this, we can substitute into the right side of

$$\dot{\phi} = \frac{f^2\phi'}{g^2(\phi'+1) - f^2\phi'}$$

and integrate again to find ϕ.

Check: if $fg = 1$

$$\phi' = \frac{1}{2(g^2 - f^2)}\left(-(g^2+f^2) \pm (g^2+f^2)\right) = 0$$

$$\dot{\phi} = 0$$

Therefore, in general we can find ϕ so that the metric takes the form:

$$ds^2 = -F(R,T)dt^2 + \frac{1}{F(R,T)}dR^2 + R^2 d\Omega^2$$

Maple gave a Bach equation of the form:

$$\frac{\partial^4 f}{\partial t^3 \partial r} = \frac{1}{2}\frac{1}{f^3r^3}\left\{-2f^5r^3\left(\frac{\partial^4 f}{\partial r^3\partial t}\right) - f^4r^3\left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial^3 f}{\partial r^3}\right) + f^4r^3\left(\frac{\partial f}{\partial r}\right)\left(\frac{\partial^3 f}{\partial r^2\partial t}\right)\right\}$$

$$+\frac{1}{2}\frac{1}{f^3r^3}\left\{-2f^5r^2\left(\frac{\partial^3 f}{\partial r^2\partial t}\right) - 2f^4r^2\left(\frac{\partial f}{\partial r}\right)\left(\frac{\partial^2 f}{\partial t\partial r}\right) + 4f^5r\left(\frac{\partial^2 f}{\partial t\partial r}\right)\left(\frac{\partial^3 f}{\partial r^2\partial t}\right)\right\}$$

$$+\frac{1}{2}\frac{1}{f^3r^3}\left\{2f^4r\left(\frac{\partial f}{\partial r}\right)\left(\frac{\partial f}{\partial t}\right) + 11f^2r^3\left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial^3 f}{\partial t^2\partial r}\right) + 5f^2r^3\left(\frac{\partial^3 f}{\partial t^3}\right)\left(\frac{\partial f}{\partial r}\right)\right\}$$

$$+\frac{1}{2}\frac{1}{f^3r^3}\left\{12f^2r^3\left(\frac{\partial^2 f}{\partial t\partial r}\right)\left(\frac{\partial^2 f}{\partial t^2}\right) - 32fr^3\left(\frac{\partial^2 f}{\partial t\partial r}\right)\left(\frac{\partial f}{\partial t}\right)^2 - 40fr^3\left(\frac{\partial^2 f}{\partial t^2}\right)\left(\frac{\partial f}{\partial r}\right)\left(\frac{\partial f}{\partial t}\right)\right\}$$

$$+\frac{1}{2}\frac{1}{f^3r^3}\left\{48r^3\left(\frac{\partial f}{\partial r}\right)\left(\frac{\partial f}{\partial t}\right)^3 - 4f^5\left(\frac{\partial^2 f}{\partial t\partial r}\right)\left(\frac{\partial f}{\partial t}\right) - 6f^3r^2\left(\frac{\partial^3 f}{\partial t^3}\right)\right\}$$

$$+\frac{1}{2}\frac{1}{f^3r^3}\left\{34f^2r^2\left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial^2 f}{\partial t^2}\right) - 32fr^2\left(\frac{\partial f}{\partial t}\right)^3\right\} \qquad (62)$$

A general solution of the form $f(r,t) = f(r)f(t)$ was obtained from maple but conversely as expected equation (61) also solves the fourth order equation. This proves Birkhoff theorem in

Weyl gravity. A different approach of proving this theorem was pursued by Riegert in [17] where he used the symmetry from the Killing vectors and conformal Killing vectors.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.1 SCHWARZSCHILD-LIKE SOLUTIONS IN WEYL GRAVITY

Using the static spherically symmetric metric,

$$ds^2 = -f^2(r)dt^2 + g^2(r)dr^2 + r^2(r)d\theta + r^2 sin^2\theta(r)d\varphi^2$$

$$g_{ab} = \begin{bmatrix} -f^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{bmatrix}$$

$$g^{a\,b} = \begin{bmatrix} -f^{-2} & 0 & 0 & 0 \\ 0 & g^{-2} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2} cosec^2\theta \end{bmatrix}$$

We found the general Schwarzschild solution from general relativity given as

$$\Rightarrow ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2 d\Omega^2$$

We then took a step further to study the behavior of such metric in Weyl gravity by performing a conformal transformation on the metric to give

$$ds^2 = -F(R)dt^2 + \frac{1}{F(R)}dR^2 + R^2 d\Omega^2$$

This gave a solution in the form

$$f = c_3 r^2 + c_1 + \frac{c_1}{r} + \frac{(c_1 - 1)(c_1 + 1)}{3c_2 r^2} \qquad (61)$$

The solution we obtained has the same form as the Schwarzschild solution with additional terms. This is in concord with the results also obtained by Bach in [1].

The coefficient with the second power of r describes the cosmological constant and the last inverse square term is a correction term possibly ascribable to the perihelion of mercury.

4.2 BIRKHOFF'S THEOREM ANALYSIS IN WEYL GRAVITY

Using the static spherically symmetric metric,

$$ds^2 = -f^2(r)dt^2 + g^2(r)dr^2 + r^2(r)d\theta + r^2 sin^2\theta(r)d\varphi^2$$

$$g_{ab} = \begin{bmatrix} -f^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{bmatrix}$$

$$g^{a\,b} = \begin{bmatrix} -f^{-2} & 0 & 0 & 0 \\ 0 & g^{-2} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2} cosec^2\theta \end{bmatrix}$$

We found the general Schwarzschild solution from general relativity given as

$$\Rightarrow ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2 d\Omega^2$$

We then showed that when we introduced time dependence to the metric

$$ds^2 = -f^2(r,t)dt^2 + g^2(r,t)dr^2 + r^2(r)d\theta + r^2 sin^2\theta(r)d\varphi^2$$

The same result is obtained, which proves the Birkhoff's theorem in general relativity.

We then took a step further to test the theorem in Weyl gravity by performing a conformal transformation on the metric to give it a new form as follows:

$$ds^2 = -F(R)dt^2 + \frac{1}{F(R)}dR^2 + R^2 d\Omega^2$$

This gave a solution in the form:

$$f = c_3 r^2 + c_1 + \frac{c_1}{r} + \frac{(c_1 - 1)(c_1 + 1)}{3c_2 r^2}.$$

as earlier explained.

We then introduced time dependence into the metric and found out that with a conformal transformation it could still take the form:

$$ds^2 = -F(R,T)dt^2 + \frac{1}{F(R,T)}dR^2 + R^2 d\Omega^2.$$

Getting solutions to this time-dependent metric was no easy task. A general solution of the form $f(r,t) = f(r)f(t)$ was obtained from maple but conversely as expected equation (61) also solves the fourth order equation (60). This proves Birkhoff theorem in Weyl gravity.

Relevant results to this theorem in Weyl gravity have been elaborately discussed by Riegert in. In his paper, Riegert was able to prove, using gauge theory, that all known field theories which admit a Birkhoff theorem, e.g. , Einstein and conformal gravity, non-Abelian gauge fields and certain $R+R^2+T^2$ theories involving both curvature and torsion, are precisely those which lack linearized spin-0 excitations [17].

4.3 THE CONFORMAL KILLING VECTORS

At this stage we directed our focus into extracting more information from the static, spherically symmetric solution due to Bach as obtained in (61). We found the conformal killing vectors to be

$-\partial t,$

$$\frac{\sqrt{-cos2\theta + 1}}{sin\theta}\partial\phi,$$

$$\frac{\sqrt{-cos2\theta + cos\phi}}{sin\theta}\partial\theta,$$

$$\frac{\sqrt{-cos2\theta + sin\phi cos\theta}}{\sin^2\theta}\partial\phi,$$

$$\frac{\sqrt{-cos2\theta + sin\phi}}{sin\theta}\partial\theta,$$

$$\frac{\sqrt{-cos2\theta + cos\phi cos\theta}}{\sin^2\theta}\partial\phi,$$

4.4 THE GEODESIC EQUATION

Using the Maple software, and parameterizing in terms of (τ), we obtained for (61), a geodesic equation with the ∂r , $\partial\theta$ and $\partial\phi$ terms given in the form

$$\left[\left(\frac{d}{d\tau}t(\tau)\right)\left(\frac{d}{d\tau}r(\tau)\right)\frac{6bcr^3(\tau) + a^2r^2(\tau) - 3c^2 - r^2(\tau)}{3bcr^3(\tau) + a^2r^2(\tau)}\right] = 0 \tag{63}$$

$$\frac{1}{r(\tau)}\left[-r(\tau)\left(\frac{d}{d\tau}\phi(\tau)\right)^2 sin\theta(\tau)cos\theta(\tau) + 2\left(\frac{d}{d\tau}r(\tau)\right)\left(\frac{d}{d\tau}\theta(\tau)\right)\right]$$

$$+\left(\frac{d^2}{d\tau^2}\theta(\tau)\right) = 0 \tag{64}$$

$$\left[r(\tau)\left(\frac{d^2}{d\tau^2}\phi(\tau)\right)sin\theta(\tau) + 2\left(\frac{d}{d\tau}r(\tau)\right)\left(\frac{d}{d\tau}\phi(\tau)\right)sin\theta(\tau)\right] = 0 \tag{65}$$

The ∂r term was ignored because it can be derived from (63), (64) and (65) respectively along with the line element due to the following initial conditions chosen.

$$\theta_0 = \frac{\pi}{2},$$

$$\phi_0 = 0,$$

$$\frac{d}{d\tau}\theta_0(\tau) = 0,$$

$$\frac{d}{d\tau}\phi_0(\tau) = 0,$$

Let us define the following:

$$\left(\frac{d}{d\tau}r(\tau)\right) \equiv u^r 0,$$

$$\left(\frac{d}{d\tau}\phi(\tau)\right) \equiv u^\phi,$$

$$\left(\frac{d}{d\tau}\theta(\tau)\right) \equiv u^\theta,$$

Therefore equation (64) alongside the initial conditions show that $\frac{d^2}{d\tau^2}\theta(\tau) = 0$

Solving (65) we have;

$$\left[r(\tau)\left(\frac{d^2}{d\tau^2}\phi(\tau)\right)sin\theta(\tau) + 2\left(\frac{d}{d\tau}r(\tau)\right)\left(\frac{d}{d\tau}\phi(\tau)\right)sin\theta(\tau)\right]\partial\phi = 0,$$

$$\Rightarrow r(\tau)\frac{d}{d\tau}\left(u^\phi\right) + 2u^r u^\phi = 0,$$

$$\Longrightarrow \frac{1}{u^{\phi}}\frac{d}{d\tau}\left(u^{\phi}\right) = -\frac{2}{r}\left(\frac{dr}{d\tau}\right),$$

Multiplying through by $d\tau$ and integrating both sides we get;

$$ln\left(\frac{u^{\phi}}{u_0^{\phi}}\right) = -2ln\left(\frac{r}{r_0}\right),$$

$$\Longrightarrow ln\left(\frac{u^{\phi}}{u_0^{\phi}}\frac{r^2}{r_0^2}\right) = 0,$$

$$\Longrightarrow \frac{u^{\phi}}{u_0^{\phi}}\frac{r^2}{r_0^2} = 1 \; or \; u^{\phi}r^2 = u_0^{\phi}r_0^2 \equiv L,$$

$$\Longrightarrow \left(\frac{d}{d\tau}\phi(\tau)\right)r^2(\tau) \equiv L. \qquad (66)$$

where L is the angular momentum.

For (63), we get the simplified geodesic equation in the form;

$$-1 = -\frac{1}{3}\frac{r(\tau)K^2}{(3bcr^3(\tau) + 3c^2r(\tau) + 3acr(\tau) + a^2 - 1)c}$$

$$+\frac{3cr(\tau)\left(\frac{d}{d\tau}r(\tau)\right)^2}{3bcr^3(\tau) + 3c^2r(\tau) + 3acr(\tau) + a^2 - 1} + \frac{L^2}{r^2(\tau)} \qquad (67)$$

Where a, b, c, and K are arbitrary constants.

4.5 THE EFFECTIVE POTENTIAL AND KEPLERIAN ORBITS

Making use of the maple program we also obtained an energy equation in the form;

$$\left(\frac{d}{d\tau}r(\tau)\right)^2 = -br^2 - cr + \frac{1}{9}\left(\frac{-3L^2c^2b - 9ac^2 + K^2}{c^2}\right) + \frac{1}{9}\left(\frac{-9L^2c^3 - 3ca^2 + 3c}{c^2r}\right)$$

$$-\frac{L^2a}{r^2} + \frac{1}{9}\left(\frac{-3L^2a^2c + 3L^2c}{c^2r^3}\right) \tag{68}$$

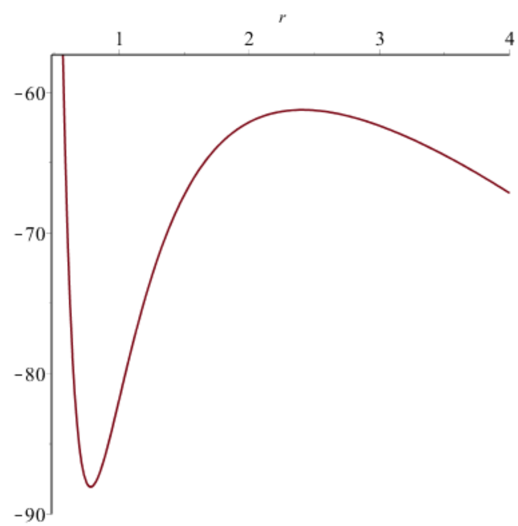
(68) can be simplified further to give:

$$\left(\frac{d}{d\tau}r(\tau)\right)^2 = kr^2 - \gamma r + L^2k + K^2 + \ 3\beta\gamma - 1 + \left(\frac{-L^2\gamma - 3\beta^2 + 2\beta}{r}\right)$$

$$+\frac{3L^2\beta\gamma - L^2}{r^2} + \left(\frac{-3L^2\beta^2\gamma + 2L^2\beta}{r^3}\right). \tag{69}$$

When plotted with certain parameters for the constants, this gives a curve of the form

```
> plot(-subs({k=0.7,gamma=0.89997,beta=1,L=10,K=1},Potentialenergy),r=0.5..4);
```




```
>
```

Figure 4.1: potential curve with adjusted parameters for k=0.7, K=1, L=10, β=1, γ=0.89997

A different parametrization gives a different curve shown below

```
> plot(-subs({k=1,gamma=10,beta=1,L=10,K=1},Potentialenergy),r=0.5..4);
```

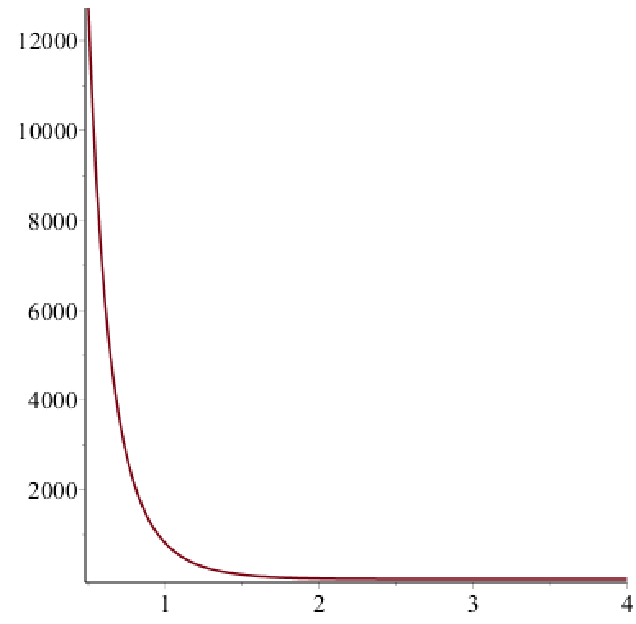


Figure 4.2: potential curve with adjusted parameters for k=1, K=1, L=10, β=1, γ=10

Both figures 4.1 and 4.2 are compatible with the minisuperspace potential describing the minisuperspace model for a Hawking black hole fully described in [23] and given under certain assumptions as:

$$V(x) = 48\exp\left[-2\sqrt{3}x\right] - [\eta x - c]^2 \tag{70}$$

Similar curves plotted in the paper are shown in figure 4.3.

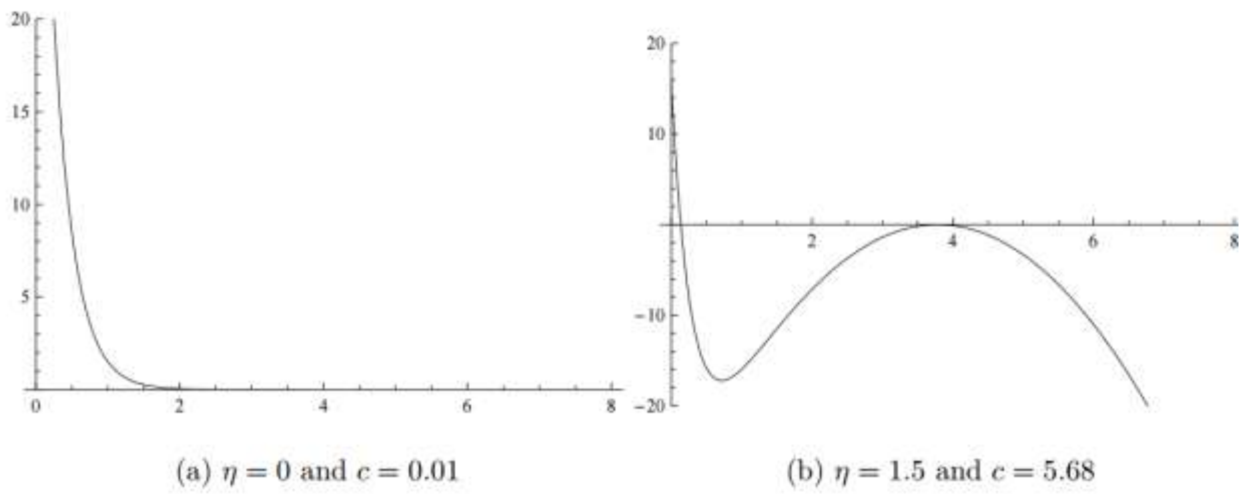


Figure 4.3: potential function for some typical values of c and η;(a) The potential for the noncommutative case, $\theta \neq 0$ and $\eta = 0$ and (b)The potential function with $\eta \neq 0$ [23]

A different parametrization gives a different curve shown below

```
> plot(-subs({k=-1500,gamma=-1000,beta=-5,L=10,K=-100},Potentialenergy),r=0.5..4);
```

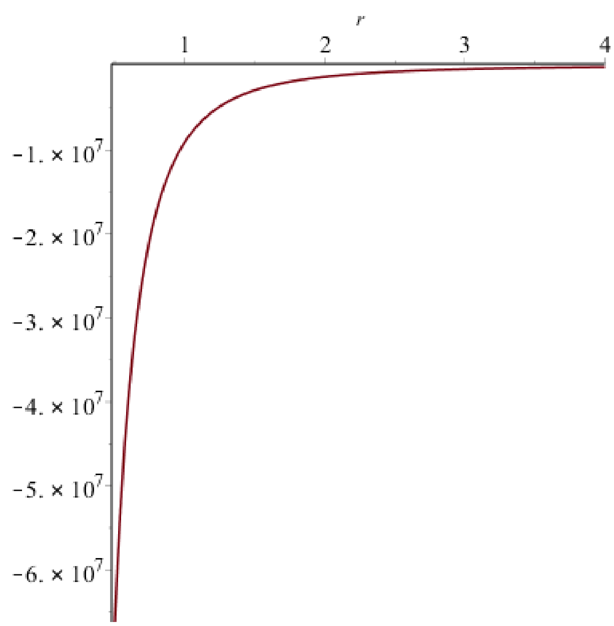


Figure 4.4: potential curve with adjusted parameters for k=-1500, K=-100, L=10, β= -5, γ= -1000

The above potential curve in figure 4.4 also matches with the effective near horizon gravitational potential given in [24] by the interaction,

$$V_{eff}(r;\omega,\alpha_{l,D})\,\underbrace{(\mathcal{H})} - \left(\Theta^2 + \frac{1}{4}\right)\frac{1}{x^2} + \frac{\alpha_{l,D}}{2k{r^2}_+}\frac{1}{x} \tag{71}$$

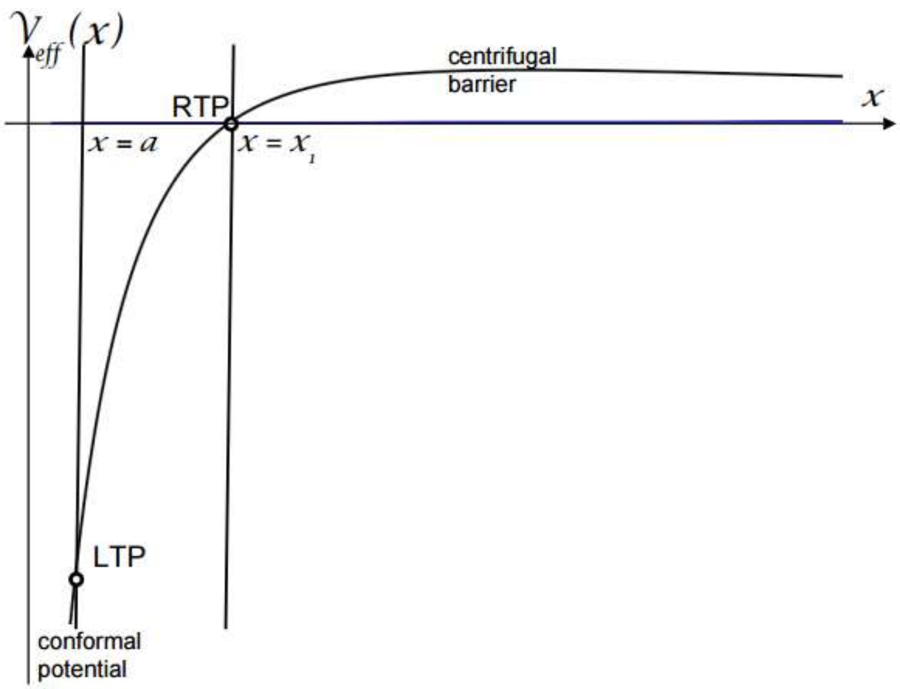


Figure 4.5

Adjusting these parameters a little bit more gives a more familiar potential curve shown in the figure below

```
plot(-subs({k=1,gamma=1,beta=1,L=10,K=10},Potentialenergy),r=0.5..4);
```

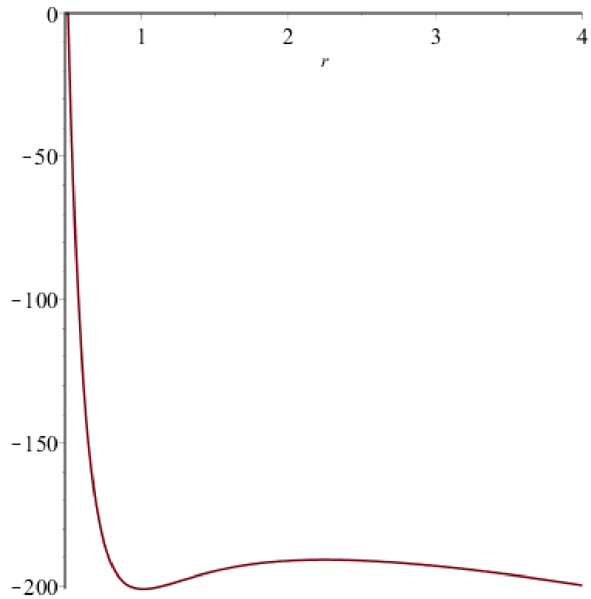


Figure 4.6: potential curve with adjusted parameters for k=1, K=10, L=10, β= 1, γ= 1

The figure above is very much comparable with the known central potential for Keplerian orbits which is clearly shown in the figure below

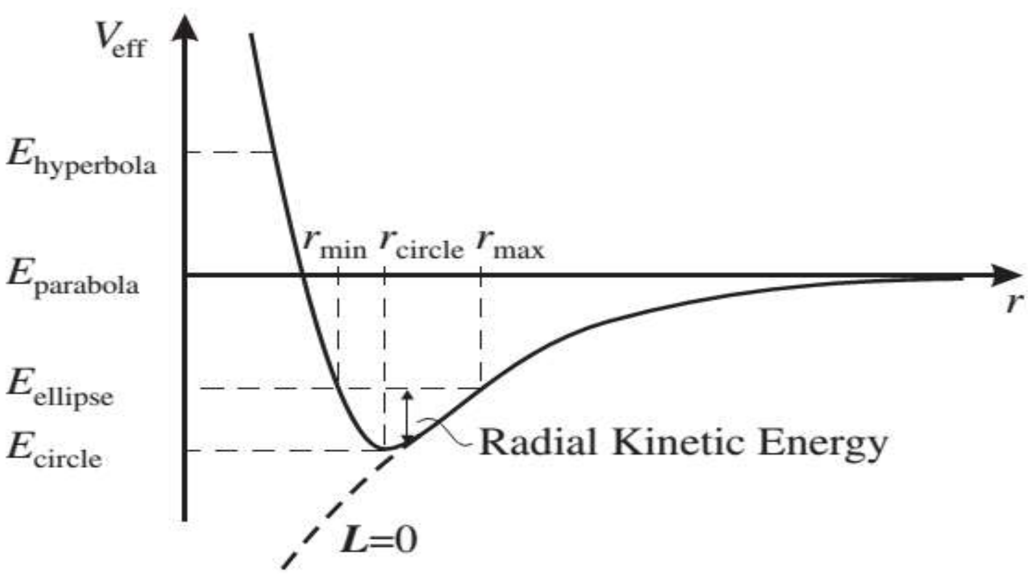


Figure 4.7: The equivalent one-dimensional potential curve for an attractive inverse square law

This describes all known motion for inverse square force laws for attractive potentials. The boundaries for energy values that give rise to hyperbolic, parabolic, elliptic, and circular orbits are as shown. For the elliptic energy values the planets are in a bounded orbit.

4.6 THE NOETHER PROBLEM AND CONSERVED QUANTITIES IN WEYL'S CONFORMAL GRAVITY

According to Noether's theorem, once there is a symmetry then there exists a conservation law, and ipso facto there is a conserved quantity. What is the quantity conserved corresponding to conformal symmetry?

The typical method for finding a conserved quantity from a symmetry is as follows:

1. Perform a general variation of the action, with no simplifying assumptions – do not yet assume the action vanishes or that the field equations hold, do not discard any surface terms.
2. Impose the field equation.
3. Restrict the variation to the symmetry variation.

What remains after this procedure will be a conservation law.

Now consider the Weyl Gravity action,

$$S = -\alpha \int C^{\alpha}{}_{\beta\mu\nu} C^{\beta}{}_{\alpha\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g}\, d^4x \tag{72}$$

In the standard treatment of Weyl gravity, this is treated as a functional of the metric $g_{\mu\nu}$ only.

While there are 15 independent conformal transformations — Lorentz transformations, translations, special conformal transformations and dilations – each result in at most a rescaling of the metric

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = e^{2\varphi} g_{\mu\nu} \tag{73}$$

Or infinitesimally,

$$\delta g_{\mu\nu} = \tilde{g}_{\mu\nu} - g_{\mu\nu} = 2 g_{\mu\nu} \delta\varphi \tag{74}$$

What is computed above is the effect of this form of transformation. It is also possible to consider each of the 15 transformations separately and a general coordinate transformation will change the metric according to,

$$\delta g_{\alpha\beta} = h_{\alpha;\beta} + h_{\beta;\alpha} \tag{75}$$

Each of the 15 transformations may be represented by such a diffeomorphism. Since $\delta g_{\alpha\beta} = f g_{\alpha\beta}$ for some function *f*, we have the conformal killing equation,

$$h_{\alpha;\beta} + h_{\beta;\alpha} = f g_{\alpha\beta} \tag{76}$$

9 for vector fields,h_α, producing conformal transformations. However, these transformations are a subset of general coordinate transformations, all of which leave any well-behaved relativistic action invariant.

To study the effect of conformal transformations on gravity theory, we examine *field* transformations in any of the two following approaches:

1. Formulating Weyl gravity in terms of a conformal connection and looking at the gauge transformations of each component of the connection. This seems like an interesting route to explore but as shown in [11] this approach leads to general relativity rather than the Bach equation.
2. Considering the effect of a conformal transformation on the metric field, $\delta g_{\alpha\beta} = 2g_{\alpha\beta}\delta\varphi$.

We consider this below:

To find the conserved current associated with the symmetry, $\delta g_{\alpha\beta} = 2g_{\alpha\beta}\delta\varphi$, we redo the variation of Section 1.2.3, without discarding surface terms.

Writing the action as,

$$S = -\alpha \int C^{\alpha}_{\beta\mu\nu} C^{\beta}_{\alpha\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g}\, d^4x$$

We note two ways the metric enters this expression: the explicit occurrence of $g_{\mu\nu}$, its inverse, and its determinant g above, and through the dependence of the Christoffel connection on the metric, from which $C^{\alpha}_{\beta\mu\nu}$ is constructed. Varying these separately we have the two terms,

$$\delta S = -2\alpha \int \delta C^{\alpha}{}_{\beta\mu\nu} C^{\beta}{}_{\alpha\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g}\, d^4x$$

$$- \alpha \int C^{\alpha}{}_{\beta\mu\nu} C^{\beta}{}_{\alpha\rho\sigma} \left(2g^{\mu\rho} \delta^{\nu}_{\lambda} \delta^{\sigma}_{\tau} - \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} g_{\lambda\tau} \right) \sqrt{-g}\, \delta g^{\lambda\tau} d^4x$$

The integrand of the second term becomes,

$$C^{\alpha}{}_{\beta\mu\nu} C^{\beta}{}_{\alpha\rho\sigma} \left(\delta^{\mu}_{\lambda} \delta^{\sigma}_{\tau} g^{\nu\sigma} - \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} g_{\lambda\tau} \right) = -2 \left(C^{\alpha\beta\mu}{}_{\lambda} C_{\alpha\beta\mu\tau} - \frac{1}{4} g_{\lambda\tau} C^{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu} \right) = 0 \qquad (77)$$

This is a well-known identity [25, 26]. We are then left with,

$$\delta S = -2\alpha \int \delta C^{\beta}{}_{\alpha\mu\nu} C^{\alpha}{}_{\beta\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} d^4x = 0 \tag{78}$$

The Weyl curvature $C^{\beta}{}_{\alpha\mu\nu}$ can be expressed in four dimensions in terms of the Riemann curvature as,

$$C^{\beta}{}_{\alpha\mu\nu} = R^{\beta}{}_{\alpha\mu\nu} - \frac{1}{2}\left(\delta^{\beta}_{\mu} R_{\alpha\nu} - g_{\alpha\mu} R^{\beta\nu} - \delta^{\beta}_{\upsilon} R_{\alpha\mu} + g_{\alpha\nu} R^{\beta\mu}\right) + \frac{1}{6}\left(g_{\alpha\nu}\delta^{\beta}_{\mu} - g_{\alpha\mu}\delta^{\beta}_{\upsilon}\right) R \tag{79}$$

Putting (79) into (78) we get:

$$0 = -2\alpha \int \delta C^{\beta}{}_{\alpha\mu\nu} C^{\alpha}{}_{\beta\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} d^4x$$

$$= -2\alpha \int \delta \left[R^{\beta}{}_{\alpha\mu\nu} + \frac{1}{2}\left(g_{\alpha\mu} R^{\beta}{}_{\upsilon} - g_{\alpha\nu} R^{\beta}{}_{\mu} - \delta^{\beta}_{\mu} R_{\alpha\nu} + \delta^{\beta}_{\upsilon} R_{\alpha\mu}\right) \right.$$
$$\left. - \frac{1}{6}\left(g_{\alpha\mu}\delta^{\beta}_{\upsilon} - g_{\alpha\upsilon}\delta^{\beta}_{\mu}\right) R\right] C^{\alpha}{}_{\beta\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} d^4x$$

$$= -2\alpha \int \left[\delta R^{\beta}{}_{\alpha\mu\nu} + \frac{1}{2}\left(g_{\alpha\mu} \delta R^{\beta}{}_{\upsilon} - g_{\alpha\nu} \delta R^{\beta}{}_{\mu} - \delta^{\beta}_{\mu} \delta R_{\alpha\nu} + \delta^{\beta}_{\upsilon} \delta R_{\alpha\mu}\right) \right.$$
$$\left. - \frac{1}{6}\left(g_{\alpha\mu}\delta^{\beta}_{\upsilon} - g_{\alpha\upsilon}\delta^{\beta}_{\mu}\right) \delta R\right] C^{\alpha}{}_{\beta}{}^{\mu\nu} \sqrt{-g} d^4x$$
$$+ 2\alpha \int \left[-\frac{1}{2}\left(\delta g_{\alpha\mu} R^{\beta}{}_{\upsilon} - \delta g_{\alpha\nu} R^{\beta}{}_{\mu}\right) + \frac{1}{6}\left(\delta g_{\alpha\mu}\delta^{\beta}_{\upsilon} - \delta g_{\alpha\upsilon}\delta^{\beta}_{\mu}\right) R\right] C^{\alpha}{}_{\beta}{}^{\mu\nu} \sqrt{-g} d^4x$$

$$= -2\alpha \int \left[C^{\alpha}{}_{\beta}{}^{\mu\nu} \delta R^{\beta}{}_{\alpha\mu\nu} + \frac{1}{2}\left(\delta g_{\alpha\mu} R^{\beta}{}_{\upsilon} - \delta g_{\alpha\nu} R^{\beta}{}_{\mu}\right) C^{\alpha}{}_{\beta}{}^{\mu\nu}\right] \sqrt{-g} d^4x$$

We carry out the variation of the curvature in two steps. First, vary the connection.

$$\delta R^{\beta}{}_{\alpha\mu\nu} = D_{\nu} \delta \Gamma^{\beta}{}_{\alpha\mu} - D_{\mu} \delta \Gamma^{\beta}{}_{\alpha\nu} \tag{80}$$

And then replacing $\delta\Gamma^{\beta}{}_{\alpha\mu}$ with its dependence on $\delta g_{\mu\upsilon}$. This first leads to:

$$= -2\alpha \int \left[C^{\alpha}{}_{\beta}{}^{\mu\nu}\left(D_{\nu}\delta\Gamma^{\beta}{}_{\alpha\mu} - D_{\mu}\delta\Gamma^{\beta}{}_{\alpha\nu}\right) + \frac{1}{2}\left(\delta g_{\alpha\mu}R^{\beta}{}_{\upsilon} - \delta g_{\alpha\nu}R^{\beta}{}_{\mu}\right)C^{\alpha}{}_{\beta}{}^{\mu\nu}\right]\sqrt{-g}d^4x$$

$$= 2\alpha \int \left[D_{\nu}\left(C^{\alpha}{}_{\beta}{}^{\mu\nu}\delta\Gamma^{\beta}{}_{\alpha\mu}\right) - D_{\mu}\left(C^{\alpha}{}_{\beta}{}^{\mu\nu}\Gamma^{\beta}{}_{\alpha\nu}\right)\right]\sqrt{-g}d^4x$$

$$+2\alpha \int \left[\left(2D_{\nu}C^{\alpha}{}_{\beta}{}^{\mu\nu}\delta\Gamma^{\beta}{}_{\alpha\mu} - \delta g_{\alpha\mu}R^{\beta}{}_{\upsilon}C^{\alpha}{}_{\beta}{}^{\mu\nu}\right)\right]\sqrt{-g}d^4x$$

$$= -2\alpha \int \left[\frac{1}{\sqrt{-g}}\partial_{\nu}\left(\sqrt{-g}C^{\alpha}{}_{\beta}{}^{\mu\nu}\delta\Gamma^{\beta}{}_{\alpha\mu}\right) - \frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}C^{\alpha}{}_{\beta}{}^{\mu\nu}\Gamma^{\beta}{}_{\alpha\nu}\right)\right]\sqrt{-g}d^4x$$

$$+2\alpha \int \left[\left(2D_{\nu}C^{\alpha}{}_{\beta}{}^{\mu\nu}\delta\Gamma^{\beta}{}_{\alpha\mu} - \delta g_{\alpha\mu}R^{\beta}{}_{\upsilon}C^{\alpha}{}_{\beta}{}^{\mu\nu}\right)\right]\sqrt{-g}d^4x \qquad (81)$$

We now integrate by parts keeping track of the surface terms. They take the form of a pure covariant divergence, which using the divergence theorem may be written as,

$$\int_{V^4} D_{\mu}J^{\mu}\sqrt{-g}d^4x = \int_{V^4} \frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}J^{\mu}\right)\sqrt{-g}d^4x$$

$$= \int_{V^4} \partial_{\mu}\left(\sqrt{-g}J^{\mu}\right)d^4x = \oint_{V^3} n_{\mu}J^{\mu}\sqrt{|g_3|}d^3x$$

Where $J^{\nu} = -2\alpha C^{\alpha}{}_{\beta}{}^{\mu\nu}\delta\Gamma^{\beta}{}_{\alpha\mu}$, the volume element $\sqrt{|g_3|}$ is $\sqrt{-g}$ evaluated on the boundary of V^4. Therefore, the variation becomes:

$$\delta s = -4\alpha \oint_{V^3} n_{\nu}C^{\alpha}{}_{\beta}{}^{\mu\nu}\delta\Gamma^{\beta}{}_{\alpha\mu}\sqrt{|g_3|}d^3x$$

$$+2\alpha \int_{V^4} \left[\left(2D_\nu C^{\alpha}{}_{\beta}{}^{\mu\nu} \delta\Gamma^{\beta}{}_{\alpha\mu} - \delta g_{\alpha\mu} R^{\beta}{}_{\upsilon} C^{\alpha}{}_{\beta}{}^{\mu\nu}\right)\right] \sqrt{-g}\, d^4x \tag{82}$$

The variation of the (Christoffel) connection with respect to the metric is;

$$\delta\Gamma^{\alpha}{}_{\mu\upsilon} = \frac{1}{2}\delta g^{\alpha\beta}\left(g_{\beta\mu,\upsilon} + g_{\beta\upsilon,\mu} - g_{\mu\upsilon,\beta}\right) + \frac{1}{2} g^{\alpha\beta}\left(\delta g_{\beta\mu,\upsilon} + \delta g_{\beta\upsilon,\mu} - \delta g_{\mu\upsilon,\beta}\right)$$

$$= \delta g^{\alpha\beta}\Gamma_{\beta\mu\nu} + \frac{1}{2} g^{\alpha\beta}\left(\delta g_{\beta\mu;\upsilon} + \delta g_{\rho\mu}\Gamma^{\rho}{}_{\beta\nu} + \delta g_{\beta\rho}\Gamma^{\rho}{}_{\mu\nu}\right)$$

$$+ \frac{1}{2} g^{\alpha\beta}\left(\delta g_{\beta\upsilon;\mu} + \delta g_{\rho\nu}\Gamma^{\rho}{}_{\beta\mu} + \delta g_{\beta\rho}\Gamma^{\rho}{}_{\nu\mu}\right)$$

$$- \frac{1}{2} g^{\alpha\beta}\left(\delta g_{\mu\nu;\beta} + \delta g_{\rho\nu}\Gamma^{\rho}{}_{\mu\beta} + \delta g_{\mu\rho}\Gamma^{\rho}{}_{\nu\beta}\right)$$

$$= \frac{1}{2} g^{\alpha\beta}\left(\delta g_{\beta\mu;\upsilon} + \delta g_{\beta\upsilon;\mu} - \delta g_{\mu\nu;\beta}\right) + \delta g^{\alpha\beta}\Gamma_{\beta\mu\nu} + \frac{1}{2} g^{\alpha\beta}\delta g_{\beta\rho}\Gamma^{\rho}{}_{\nu\mu} + \frac{1}{2} g^{\alpha\beta}\delta g_{\beta\rho}\Gamma^{\rho}{}_{\nu\mu}$$

$$= \frac{1}{2} g^{\alpha\beta}\left(D_\upsilon \delta g_{\beta\mu} + D_\mu \delta g_{\beta\nu} - D_\beta \delta g_{\mu\nu}\right) + \delta g^{\alpha\beta}\Gamma_{\beta\mu\nu} - \frac{1}{2}\delta g^{\alpha\beta}\Gamma_{\beta\mu\nu} - \frac{1}{2}\delta g^{\alpha\beta}\Gamma_{\beta\nu\mu}$$

$$= \frac{1}{2} g^{\alpha\beta}\left(D_\upsilon \delta g_{\beta\mu} + D_\mu \delta g_{\beta\nu} - D_\beta \delta g_{\mu\nu}\right) \tag{83}$$

Substituting, we again perform several integrations by parts, and get additional surface terms. We now consider the terms one at a time.

The first integral of equation (82) becomes:

$$-4\alpha \oint_{V^3} n_\nu C^{\alpha}{}_{\beta}{}^{\mu\nu} \delta\Gamma^{\beta}{}_{\alpha\mu} \sqrt{|g_3|}\, d^3x$$

$$= -4\alpha \oint_{V^3} n_\nu C^{\alpha}{}_{\beta}{}^{\mu\nu} \left(\frac{1}{2} g^{\beta\rho}\left(D_\mu \delta g_{\rho\alpha} + D_\alpha \delta g_{\rho\mu} - D_\rho \delta g_{\alpha\mu}\right)\right)\sqrt{|g_3|}d^3x$$

$$= -2\alpha \oint_{V^3} n_\nu C^{\alpha\rho\mu\nu}\left(D_\mu \delta g_{\rho\alpha} + D_\alpha \delta g_{\rho\mu} - D_\rho \delta g_{\alpha\mu}\right)\sqrt{|g_3|}d^3x$$

$$= -2\alpha \oint_{V^3} \left(D_\mu\left(n_\nu C^{\alpha\rho\mu\nu}\delta g_{\rho\alpha}\right) + D_\alpha\left(n_\nu C^{\alpha\rho\mu\nu}\delta g_{\rho\mu}\right) - D_\rho\left(n_\nu C^{\alpha\rho\mu\nu}\delta g_{\alpha\mu}\right)\right)\sqrt{|g_3|}d^3x$$

$$+2\alpha \oint_{V^3} \left(D_\mu\left(n_\nu C^{\alpha\rho\mu\nu}\delta g_{\rho\alpha}\right) + D_\alpha\left(n_\nu C^{\alpha\rho\mu\nu}\delta g_{\rho\mu}\right) - D_\rho\left(n_\nu C^{\alpha\rho\mu\nu}\delta g_{\alpha\mu}\right)\right)\sqrt{|g_3|}d^3x \qquad (84)$$

The integrals in the first row of (84) have the form of divergences again. Simplifying, $n_\nu C^{\alpha\rho\mu\nu}\delta g_{\rho\alpha} = 0$ by the antisymmetry of $C^{\alpha\rho\mu\nu}$ and the symmetry of $\delta g_{\rho\alpha}$, while the second and third integrals combine,

$$-2\alpha \oint_{V^3} \left(D_\alpha\left(n_\nu C^{\alpha\rho\mu\nu}\delta g_{\rho\mu}\right) - D_\rho\left(n_\nu C^{\alpha\rho\mu\nu}\delta g_{\alpha\mu}\right)\right)\sqrt{|g_3|}d^3x$$

$$= -4\alpha \oint_{V^3} D_\alpha\left(n_\nu C^{\alpha\rho\mu\nu}\delta g_{\rho\mu}\right)\sqrt{|g_3|}d^3x$$

Now we set $J^\alpha = n_\nu C^{\alpha\rho\mu\nu}\delta g_{\rho\mu}$ and the first line becomes:

$$-4\alpha \oint_{V^3} D_\alpha J^\alpha \sqrt{|g_3|}d^3x$$

We may still write the divergence as;

$$D_\alpha J^\alpha = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}J^\alpha\right)$$

but must evaluate this on the boundary. The volume element factors, $\sqrt{-g} = \sqrt{|g_1|}\sqrt{|g_3|}$ where $\sqrt{|g_1|}$ is the volume element in the n_μ direction. Choosing a unit coordinate in this direction at the boundary, we may set $g_1 = 1$. Then we decompose the current and the derivative along and perpendicular to the boundary:

$$J^\mu = n^\mu(n_\nu J^\nu) + P^{\nu}_{\downarrow\downarrow\ \nu}J^\nu$$

$$J^\mu = n^\mu\left(n_\alpha n_\nu C^{\alpha\rho\sigma\nu}\delta g_{\rho\sigma}\right) + \left(\delta^\mu_\alpha - n^\mu n_\alpha\right)n_\nu C^{\alpha\rho\sigma\nu}\delta g_{\rho\sigma} \equiv (\rho, J^t_{\downarrow\downarrow})$$

and

$$\partial_\mu = n_\mu n^\nu\partial_\nu + P^{\nu}_{\downarrow\downarrow\ \mu}\partial_\nu = \left(\frac{\partial}{\partial n}, \partial^{\downarrow\downarrow}_t\right)$$

Thus,

$$D_\mu J^\mu = \frac{1}{\sqrt{|g_3|}}\left(\sqrt{|g_3|}\frac{\partial\rho}{\partial n} + \partial^{\downarrow\downarrow}_t\left(\sqrt{|g_3|}J^t_{\downarrow\downarrow}\right)\right) = \frac{\partial\rho}{\partial n} + \frac{1}{\sqrt{|g_3|}}\partial^{\downarrow\downarrow}_t\left(\sqrt{|g_3|}\,J^t_{\downarrow\downarrow}\right)$$

The first line of (82) becomes:

$$-4\alpha\oint_{V^3} D_\alpha J^\alpha\sqrt{|g_3|}d^3x = -4\alpha\oint_{V^3}\frac{\partial\rho}{\partial n}\sqrt{|g_3|}d^3x - 4\alpha\oint_{V^3}\frac{1}{\sqrt{|g_3|}}\partial^{\downarrow\downarrow}_t\left(\sqrt{|g_3|}\,J^t_{\downarrow\downarrow}\right)\sqrt{|g_3|}d^3x$$

$$= -4\alpha\oint_{V^3}\frac{\partial\rho}{\partial n}\sqrt{|g_3|}d^3x - 4\alpha\oint_{V^3}\partial^{\downarrow\downarrow}_t\left(\sqrt{|g_3|}\,J^t_{\downarrow\downarrow}\right)d^3x$$

The second integral now yields to the divergence theorem, giving the normal component of the integrand evaluated on the boundary of V^3. However, the boundary of a region is closed, and itself has no boundary so this term vanishes identically. For the first term we get the average of the outward normal derivative of the normal component of the current over the boundary:

$$-4\alpha \langle\frac{\partial\rho}{\partial n}\rangle_{V^3} = -4\alpha \oint_{V^3} \frac{\partial\rho}{\partial n}\sqrt{|g_3|}d^3x = -4\alpha \oint_{V^3} \frac{\partial}{\partial n}\left(n_\alpha n_\nu C^{\alpha\rho\sigma\nu}\delta g_{\rho\sigma}\right)\sqrt{|g_3|}d^3x$$

This is what remains of the first line.

The second line of integrals in (84) also combines into a single term, with the first integral vanishing by symmetry and the remaining two combining,

$$+4\alpha \oint_{V^3} D_\alpha\left(n_\nu C^{\alpha\rho\mu\nu}\delta g_{\rho\mu}\right)\sqrt{|g_3|}d^3x$$

Combining the remaining terms of (84),

$$-4\alpha \oint_{V^3} n_\nu C^{\alpha}{}_{\beta}{}^{\mu\nu}\delta\Gamma^{\beta}{}_{\alpha\mu}\sqrt{|g_3|}d^3x =$$

$$-4\alpha \oint_{V^3} \frac{\partial}{\partial n}\left(n_\alpha n_\nu C^{\alpha\rho\sigma\nu}\delta g_{\rho\sigma}\right)\sqrt{|g_3|}d^3x + 4\alpha \oint_{V^3} D_\alpha\left(n_\nu C^{\alpha\rho\mu\nu}\delta g_{\rho\mu}\right)\sqrt{|g_3|}d^3x \qquad (85)$$

Now we consider the second line of (82) is given as before as:

$$+2\alpha \int_{V^4} \left[\left(2D_\nu C^{\alpha}{}_{\beta}{}^{\mu\nu}\delta\Gamma^{\beta}{}_{\alpha\mu} - \delta g_{\alpha\mu}R^{\beta}{}_{\upsilon}C^{\alpha}{}_{\beta}{}^{\mu\nu}\right)\right]\sqrt{-g}d^4x$$

the final term of which is already a metric variation. Expanding the first term,

$$4\alpha \int\limits_{V^4} \left[D_\nu C^{\alpha}{}_{\beta}{}^{\mu\nu} \delta\Gamma^{\beta}{}_{\alpha\mu}\right] \sqrt{-g} d^4x$$

$$= 4\alpha \int\limits_{V^4} \left[D_\nu C^{\alpha}{}_{\beta}{}^{\mu\nu}\right] \left(\frac{1}{2} g^{\beta\rho} \left(D_\mu \delta g_{\rho\alpha} + D_\alpha \delta g_{\rho\mu} - D_\rho \delta g_{\alpha\mu}\right)\right) \sqrt{-g} d^4x$$

$$= 2\alpha \int\limits_{V^4} \left(D_\nu C^{\alpha\rho\mu\nu} D_\mu \delta g_{\rho\alpha} + D_\nu C^{\alpha\rho\mu\nu} D_\alpha \delta g_{\rho\mu} - D_\nu C^{\alpha\rho\mu\nu} D_\rho \delta g_{\alpha\mu}\right) \sqrt{-g} d^4x$$

$$= 4\alpha \int\limits_{V^4} D_\nu C^{\alpha\rho\mu\nu} D_\alpha \delta g_{\rho\mu} \sqrt{-g} d^4x$$

$$= 4\alpha \int\limits_{V^4} D_\alpha \left(D_\nu C^{\alpha\rho\mu\nu} \delta g_{\rho\mu}\right) \sqrt{-g} d^4x - 4\alpha \int\limits_{V^4} D_\alpha D_\nu C^{\alpha\rho\mu\nu} \delta g_{\rho\mu} \sqrt{-g} d^4x$$

Using the divergence theorem on the first term,

$$D_\alpha\left(D_\nu C^{\alpha\rho\mu\nu} \delta g_{\rho\mu}\right) = \frac{1}{\sqrt{-g}} \partial_\alpha\left(\sqrt{-g} D_\nu C^{\alpha\rho\mu\nu} \delta g_{\rho\mu}\right)$$

The integral becomes:

$$4\alpha \int\limits_{V^4} D_\alpha\left(D_\nu C^{\alpha\rho\mu\nu} \delta g_{\rho\mu}\right) \sqrt{-g} d^4x = 4\alpha \int\limits_{V^4} \partial_\alpha\left(\sqrt{-g} D_\nu C^{\alpha\rho\mu\nu} \delta g_{\rho\mu}\right) d^4x$$

$$= 4\alpha \int\limits_{V^3} \left(n_\alpha D_\nu C^{\alpha\rho\mu\nu} \delta g_{\rho\mu}\right) \sqrt{|g_1|} \sqrt{|g_3|} d^3x$$

$$= 4\alpha \int_{V^3} \left(n_\alpha D_\nu C^{\alpha\rho\mu\nu} \delta g_{\rho\mu}\right)\sqrt{|g_3|}d^3x$$

The second line of (82) is therefore,

$$4\alpha \int_{V^3} \left(n_\alpha D_\nu C^{\alpha\rho\mu\nu} \delta g_{\rho\mu}\right)\sqrt{|g_3|}d^3x + 2\alpha \int_{V^4} \left(2D_\rho D_\nu C^{\alpha\rho\mu\nu} - R_{\beta\nu} C^{\alpha\beta\mu\nu}\right) \delta g_{\alpha\mu}\sqrt{-g}d^4x \quad (86)$$

Now combining (85) and (86), we have the general metric variation as,

$$\delta S = -4\alpha \oint_{V^3} \frac{\partial}{\partial n}\left(n_\alpha n_\nu C^{\alpha\rho\sigma\nu} \delta g_{\rho\sigma}\right)\sqrt{|g_3|}d^3x + 4\alpha \oint_{V^3} D_\alpha\left(n_\nu C^{\alpha\rho\mu\nu} \delta g_{\rho\mu}\right)\sqrt{|g_3|}d^3x$$

$$+4\alpha \int_{V^3} \left(n_\alpha D_\nu C^{\alpha\rho\mu\nu} \delta g_{\rho\mu}\right)\sqrt{|g_3|}d^3x + 2\alpha \int_{V^4} \left(2D_\rho D_\nu C^{\alpha\rho\mu\nu} - R_{\beta\nu} C^{\alpha\beta\mu\nu}\right) \delta g_{\alpha\mu}\sqrt{-g}d^4x$$

$$= 4\alpha \int_{V^3} \left(-\frac{\partial}{\partial n}\left(n_\alpha n_\nu C^{\alpha\rho\sigma\nu} \delta g_{\rho\sigma}\right) + D_\alpha\left(n_\nu C^{\alpha\rho\mu\nu} \delta g_{\rho\mu}\right) + n_\alpha D_\nu C^{\alpha\rho\mu\nu} \delta g_{\rho\mu}\right)\sqrt{|g_3|}d^3x$$

$$+2\alpha \int_{V^4} \left(2D_\rho D_\nu C^{\alpha\rho\mu\nu} - R_{\beta\nu} C^{\alpha\beta\mu\nu}\right) \delta g_{\alpha\mu}\sqrt{-g}d^4x \qquad (87)$$

Now we may apply the rest of the Noether procedure. Imposing the Bach equation, the final line of (87) vanishes, leaving us with

$$\delta S = 4\alpha \int_{V^3} \left(-\frac{\partial}{\partial n}\left(n_\alpha n_\nu C^{\alpha\rho\sigma\nu} \delta g_{\rho\sigma}\right) + D_\alpha\left(n_\nu C^{\alpha\rho\mu\nu} \delta g_{\rho\mu}\right) + n_\alpha D_\nu C^{\alpha\rho\mu\nu} \delta g_{\rho\mu}\right)\sqrt{|g_3|}d^3x$$

Restricting $\delta g_{\rho\rho}$ to $2g_{\rho\mu}\delta\varphi$, the variation must vanish by conformal invariance of the action, while the conserved quantity is:

$$\Sigma = 8\alpha \int\limits_{V^3} \left(-\frac{\partial}{\partial n}\left(n_\alpha n_\nu C^{\alpha\rho\sigma\nu} g_{\rho\sigma}\delta\varphi\right) + D_\alpha\left(n_\nu C^{\alpha\rho\mu\nu} g_{\rho\mu}\delta\varphi\right) + n_\alpha D_\nu C^{\alpha\rho\mu\nu} g_{\rho\mu}\delta\varphi\right)\sqrt{|g_3|}d^3x$$

However the tracelessness of the Conformal tensor makes this quantity vanish since in the first term $C^{\alpha\rho\sigma\nu}g_{\rho\sigma} = 0$, in the second term $D_\alpha(n_\nu C^{\alpha\rho\mu\nu})g_{\rho\mu} = D_\alpha(n_\nu C^{\alpha\rho\mu\nu})g_{\rho\mu} = 0$, and in the last $n_\alpha D_\nu C^{\alpha\rho\mu\nu} g_{\rho\mu} = n_\alpha D_\nu C^{\alpha\rho\mu\nu} g_{\rho\mu} = 0$.

Every term vanishes identically, since the metrics commute with covariant derivatives to give traces of the traceless Weyl curvature. The conserved quantity vanishes identically. This is unusual, but not forbidden.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

In this work, we have reviewed the concept of gravity according to Einstein in general relativity and also according to Weyl in conformal gravity. We verified that Birkhoff's theorem holds for both cases. We also verified the fact that all metrics which are solutions to Einstein vacuum equations are also solutions to the Bach equation but not all that are solutions to the Bach equations are solutions to the Einstein equation. We gave example of such metrics with proofs. We also found out that we can also get Schwarzschild-like solutions for the Bach equation under certain conformal transformation on the spherically symmetric static metric with a cosmological constant.

Furthermore, we deduced the geodesic equation and found the corresponding energy equation. The effective potential curve from Weyl gravity was seen to take the same shape as several other known potential curves under certain parametrization. This includes that of the central potential of the Keplerian orbits as deduced from the Schwarzschild solution in General relativity. The effective Horizon potential curve was also found alongside the potential for the minisuperspace model.

We also took a step further to determine conserved quantities in Weyl gravity and found out that the conserved current always vanishes identically [7].

Given that Weyl gravity has more solutions to it than Einstein gravity, it deserves more attention and study in order to find possible solutions to certain puzzles in general relativity such as quantization, dark matter and dark energy as proposed also in [3,11].

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APPENDIX I

DETAILED PROOF OF BIRKHOFF THEOREM IN GENERAL RELATIVITY

We begin with the general static spherically symmetric line-element of the form given in (48), (49), and (50). We rewrite them as follows;

$$ds^2 = -f^2(r)dt^2 + g^2(r)dr^2 + r^2(r)d\theta + r^2 sin^2\theta(r)d\varphi^2 \tag{88}$$

$$g_{ab} = \begin{bmatrix} -f^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{bmatrix} \tag{89}$$

$$g^{a\,b} = \begin{bmatrix} -f^{-2} & 0 & 0 & 0 \\ 0 & g^{-2} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2} cosec^2\theta \end{bmatrix} \tag{90}$$

We aim at finding the Schwarzschild Solutions from the basics by following the detailed steps shown below:

Step 1:

We first we find the non-zero components of the Affine connection or Christoffel symbol.

$$\Gamma_{\alpha\mu\nu} = \frac{1}{2}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right) \tag{91}$$

We already know that

$$\Gamma_{\alpha\mu\nu} = \Gamma_{\alpha\nu\mu} \text{ and } \Gamma_{\alpha\mu\nu} = -\Gamma_{\mu\alpha\nu} \tag{92}$$

Working out (91) with the help of (92) we get thirteen non-zero components for the affine connection:

$$\Gamma_{001} = \frac{1}{2}\left(g_{00,1} + g_{01,0} - g_{01,0}\right) = \frac{(-f^2)'}{2} = -\frac{2ff'}{2} = -ff'$$

$$\Gamma_{010} = \Gamma_{001} = -ff' and\, \Gamma_{100} = -\Gamma_{010} = ff'$$

$$\Gamma_{111} = \frac{1}{2}\left(g_{11,1} + g_{11,1} - g_{11,1}\right) = \frac{(g^2)'}{2} = \frac{2gg'}{2} = gg'$$

$$\Gamma_{221} = \frac{1}{2}\left(g_{22,1} + g_{21,2} - g_{21,2}\right) = \frac{(r^2)'}{2} = \frac{2r}{2} = r$$

$$\Gamma_{221} = \Gamma_{212} = r \;\; and\, \Gamma_{122} = -\Gamma_{212} = -r$$

$$\Gamma_{331} = \frac{1}{2}\left(g_{33,1} + g_{31,3} - g_{31,3}\right) = \frac{d}{dr}\frac{(r^2 sin^2\theta)}{2} = \frac{2r sin^2\theta}{2} = r sin^2\theta$$

$$\Gamma_{313} = \Gamma_{331} = r sin^2\theta \;\; and \;\; \Gamma_{133} = -\Gamma_{313} = -r sin^2\theta$$

$$\Gamma_{332} = \frac{1}{2}\left(g_{33,2} + g_{32,3} - g_{32,3}\right) = \frac{d}{d\theta}\frac{(r^2 sin^2\theta)}{2} = \frac{2r^2 sin\theta cos\theta}{2} = r^2 sin\theta cos\theta$$

$$\Gamma_{323} = \Gamma_{332} = r^2 sin\theta cos\theta \;\;\; and \;\; \Gamma_{233} = -\Gamma_{323} = -r^2 sin\theta cos\theta$$

The results are summarized in the table 3.1.

Step 2:

We now compute the corresponding mixed form of the affine connection

$$\Gamma^{\alpha}{}_{\mu\nu} = g^{\alpha\beta}\Gamma_{\beta\mu\nu} \tag{93}$$

We also make good use of the identities in (52).

$$\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu} \tag{94}$$

Due to the diagonal form of our metric, it becomes easy to do this and the only surviving terms are

$$\Gamma^{0}{}_{01} = g^{00}\Gamma_{001} = \left(-\frac{1}{f^2}\right)(—ff') = \frac{f'}{f} = \Gamma^{0}{}_{10}$$

$$\Gamma^{1}{}_{00} = g^{11}\Gamma_{100} = \frac{1}{g^2}(ff') = \frac{ff'}{g^2}$$

$$\Gamma^{1}{}_{11} = g^{11}\Gamma_{111} = \left(\frac{1}{g^2}\right)(gg') = \frac{g'}{g}$$

$$\Gamma^{2}{}_{21} = g^{22}\Gamma_{221} = \left(\frac{1}{r^2}\right)(r) = \frac{1}{r} = \Gamma^{2}{}_{12}$$

$$\Gamma^{1}{}_{22} = g^{11}\Gamma_{122} = \left(\frac{1}{g^2}\right)(-r) = -\frac{r}{g^2}$$

$$\Gamma^{3}{}_{31} = g^{33}\Gamma_{331} = \left(\frac{1}{r^2 sin^2\theta}\right)(rsin^2\theta) = \frac{1}{r} = \Gamma^{3}{}_{13}$$

$$\Gamma^{1}{}_{33} = g^{11}\Gamma_{133} = \left(\frac{1}{g^2}\right)(-rsin^2\theta) = -\frac{rsin^2\theta}{g^2}$$

$$\Gamma^{3}{}_{32} = g^{33}\Gamma_{332} = \left(\frac{1}{r^2 sin^2\theta}\right)(r^2 sin\theta cos\theta) = cot = \Gamma^{3}{}_{23}$$

$$\Gamma^{2}{}_{33} = g^{22}\Gamma_{233} = \left(\frac{1}{r^2}\right)(-r^2 sin\theta cos\theta) = -sin\theta cos\theta$$

The results are summarized in the table 3.2.

Step 3:

We proceed to compute the curvature.

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\rho\mu}\Gamma^{\rho}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\rho\nu}\Gamma^{\rho}{}_{\beta\mu} \tag{95}$$

$$R^{1}{}_{010} = \Gamma^{1}{}_{00,1} - \Gamma^{1}{}_{01,0} + \Gamma^{1}{}_{\rho 1}\Gamma^{\rho}{}_{00} - \Gamma^{1}{}_{\rho 0}\Gamma^{\rho}{}_{01}$$

$$= \Gamma^{1}{}_{00,1} - \Gamma^{1}{}_{01,0} + \Gamma^{1}{}_{11}\Gamma^{1}{}_{00} - \Gamma^{1}{}_{10}\Gamma^{1}{}_{01} + \Gamma^{1}{}_{01}\Gamma^{0}{}_{00} - \Gamma^{1}{}_{00}\Gamma^{0}{}_{01}$$

$$= \frac{d}{dr}\left(\frac{ff'}{g^2}\right) + \left(\frac{g'}{g}\right)\left(\frac{ff'}{g^2}\right) - \left(\frac{f'}{f}\right)\left(\frac{ff'}{g^2}\right) = \frac{d}{dr}\left(\frac{ff'}{g^2}\right) + \left(\frac{g'}{g}\right)\left(\frac{ff'}{g^2}\right) - \left(\frac{f'}{g}\right)^2$$

$$= \frac{ff''}{g^2} + \left(\frac{f'}{g}\right)^2 + ff'\left(-\frac{2g'}{g^3}\right) + \frac{g'ff'}{g^3} - \left(\frac{f'}{g}\right)^2$$

$$= \frac{ff''}{g^2} - \frac{g'ff'}{g^3} = \frac{gff'' - g'ff'}{g^3}$$

$$\Rightarrow R^{1}{}_{010} = \frac{gff'' - g'ff'}{g^3}$$

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\rho\mu}\Gamma^{\rho}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\rho\nu}\Gamma^{\rho}{}_{\beta\mu}$$

$$R^{0}{}_{101} = \Gamma^{0}{}_{11,0} - \Gamma^{0}{}_{10,1} + \Gamma^{0}{}_{\rho 0}\Gamma^{\rho}{}_{11} - \Gamma^{0}{}_{\rho 1}\Gamma^{\rho}{}_{10}$$

$$= \Gamma^{0}{}_{11,0} - \Gamma^{0}{}_{10,1} + \Gamma^{0}{}_{00}\Gamma^{0}{}_{11} - \Gamma^{0}{}_{01}\Gamma^{0}{}_{10} + \Gamma^{0}{}_{10}\Gamma^{1}{}_{11} - \Gamma^{0}{}_{11}\Gamma^{1}{}_{10}$$

$$= -\frac{d}{dr}\left(\frac{f'}{f}\right) - \left(\frac{f'}{f}\right)^2 + \left(\frac{g'}{g}\right)\left(\frac{f'}{f}\right)$$

$$= -\left(\frac{ff'' - f'f'}{f^2}\right) - \left(\frac{f'}{f}\right)^2 + \left(\frac{g'f'}{gf}\right) = -\frac{ff''}{f^2} + \left(\frac{f'}{f}\right)^2 - \left(\frac{f'}{f}\right)^2 + \left(\frac{g'f'}{gf}\right)$$

$$\Rightarrow R^0{}_{101} = -\frac{ff''}{f^2} + \left(\frac{g'f'}{gf}\right) = \frac{g'ff' - gff''}{gf^2}$$

Therefore, using the identity

$$R^\alpha{}_{\beta\mu\nu} = g^{\alpha\rho} R_{\rho\beta\mu\nu} = g^{\alpha\rho} g_{\eta\gamma} R^{\ \eta}{}_{\beta\mu\nu} \tag{96}$$

$$R^1{}_{010} = g^{11} R_{1010} = g^{11} g_{00} R^0{}_{101} = -\frac{f^2}{g^2}\left[\frac{g'ff' - gff''}{gf^2}\right]$$

$$\Rightarrow R^1{}_{010} = \frac{gff'' - g'ff'}{g^3}$$

which confirms what we got before for, $R^1{}_{010}$ which confirms the validity of the identity (96).

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu}$$

$$R^1{}_{212} = \Gamma^1{}_{22,1} - \Gamma^1{}_{21,2} + \Gamma^1{}_{\rho 1}\Gamma^\rho{}_{22} - \Gamma^1{}_{\rho 2}\Gamma^\rho{}_{21}$$

$$= \Gamma^1{}_{22,1} - \Gamma^1{}_{21,2} + \Gamma^1{}_{11}\Gamma^1{}_{22} - \Gamma^1{}_{12}\Gamma^1{}_{21} + \Gamma^1{}_{01}\Gamma^0{}_{22} - \Gamma^1{}_{02}\Gamma^0{}_{21}$$

$$+\Gamma^1{}_{21}\Gamma^2{}_{22} - \Gamma^1{}_{22}\Gamma^2{}_{21} + \Gamma^1{}_{31}\Gamma^3{}_{22} - \Gamma^1{}_{32}\Gamma^3{}_{21}$$

$$= \frac{d}{dr}\left(-\frac{r}{g^2}\right) + \left(\frac{g'}{g}\right)\left(-\frac{r}{g^2}\right) - \left(\frac{1}{r}\right)\left(-\frac{r}{g^2}\right) = -\left(\frac{g^2.1 - r.2gg'}{g^4}\right) - \left(\frac{g'r}{g^3}\right) + \left(\frac{1}{g^2}\right)$$

$$R^1{}_{212} = \frac{-g^2 + 2rgg' - gg'r + g^2}{g^4} = \frac{rgg'}{g^4}$$

$$\Rightarrow R^1{}_{212} = \frac{rg'}{g^3}$$

Therefore, using the identity in (96), we have:

$$R^{\alpha}{}_{\beta\mu\nu} = g^{\alpha\rho} R_{\rho\beta\mu\nu} = g^{\alpha\rho} g_{\eta\gamma} R^{\eta}{}_{\beta\mu\nu}$$

$$R^{2}{}_{121} = g^{22} R_{2121} = g^{22} g_{11} R^{1}{}_{212}$$

$$= \left(\frac{1}{r^2}\right)(g^2)\left(\frac{rg'}{g^3}\right) = \frac{g'}{rg}$$

$$\Rightarrow R^{2}{}_{121} = \frac{g'}{rg}$$

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\rho\mu}\Gamma^{\rho}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\rho\nu}\Gamma^{\rho}{}_{\beta\mu}$$

$$R^{1}{}_{313} = \Gamma^{1}{}_{33,1} - \Gamma^{1}{}_{31,3} + \Gamma^{1}{}_{\rho 1}\Gamma^{\rho}{}_{33} - \Gamma^{1}{}_{\rho 3}\Gamma^{\rho}{}_{31}$$

$$= \Gamma^{1}{}_{33,1} - \Gamma^{1}{}_{31,3} + \Gamma^{1}{}_{11}\Gamma^{1}{}_{33} - \Gamma^{1}{}_{13}\Gamma^{1}{}_{31} + \Gamma^{1}{}_{01}\Gamma^{0}{}_{33} - \Gamma^{1}{}_{03}\Gamma^{0}{}_{31}$$

$$+\Gamma^{1}{}_{21}\Gamma^{2}{}_{33} - \Gamma^{1}{}_{23}\Gamma^{2}{}_{31} + \Gamma^{1}{}_{31}\Gamma^{3}{}_{33} - \Gamma^{1}{}_{33}\Gamma^{3}{}_{31}$$

$$= \frac{d}{dr}\left(-\frac{r sin^2\theta}{g^2}\right) + \left(\frac{g'}{g}\right)\left(-\frac{r sin^2\theta}{g^2}\right) - \frac{r sin^2\theta}{g^2}\left(\frac{1}{r}\right)$$

$$= -\left(\frac{g^2 sin^2\theta - r sin^2\theta . 2gg'}{g^4}\right) - \left(\frac{g' r sin^2\theta}{g^3}\right) + \left(\frac{sin^2\theta}{g^2}\right)$$

$$= \left(\frac{2rgg' sin^2\theta - g^2 sin^2\theta - rgg' sin^2\theta + g^2 sin^2\theta}{g^4}\right) = \frac{rgg' sin^2\theta}{g^4}$$

$$\Rightarrow R^{1}{}_{313} = \left(\frac{rg' sin^2\theta}{g^3}\right)$$

Therefore, using the identity in (96), we have:

$$R^{\alpha}{}_{\beta\mu\nu} = g^{\alpha\rho} R_{\rho\beta\mu\nu} = g^{\alpha\rho} g_{\eta\gamma} R^{\eta}{}_{\beta\mu\nu}$$

$$R^3{}_{131} = g^{33} R_{3131} = g^{33} g_{11} R^{\ 1}{}_{313}$$

$$= \left(\frac{1}{r^2 sin^2\theta}\right) g^2 \left(\frac{rg' sin^2\theta}{g^3}\right)$$

$$\Rightarrow R^3{}_{131} = \frac{g'}{rg}$$

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu}$$

$$R^2{}_{323} = \Gamma^2{}_{33,2} - \Gamma^2{}_{32,3} + \Gamma^2{}_{\rho 2}\Gamma^\rho{}_{33} - \Gamma^2{}_{\rho 3}\Gamma^\rho{}_{32}$$

$$= \Gamma^2{}_{33,2} - \Gamma^2{}_{32,3} + \Gamma^2{}_{02}\Gamma^0{}_{33} - \Gamma^2{}_{03}\Gamma^0{}_{32} + \Gamma^2{}_{12}\Gamma^1{}_{33} - \Gamma^2{}_{13}\Gamma^1{}_{32}$$

$$+\Gamma^2{}_{22}\Gamma^2{}_{33} - \Gamma^2{}_{23}\Gamma^2{}_{32} + \Gamma^2{}_{32}\Gamma^3{}_{33} - \Gamma^2{}_{33}\Gamma^3{}_{32}$$

$$= \frac{d}{d\theta}(-sin\theta cos\theta) + \left(\frac{1}{r}\right)\left(-\frac{rsin^2\theta}{g^2}\right) - sin\theta cos\theta (cot\theta)$$

$$= sin^2\theta - cos^2\theta - \frac{sin^2\theta}{g^2} + cos^2\theta$$

$$\Rightarrow R^2{}_{323} = sin^2\theta\left(1 - \frac{1}{g^2}\right)$$

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} - \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} + \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu}$$

$$R^3{}_{232} = \Gamma^3{}_{22,3} - \Gamma^3{}_{23,2} + \Gamma^3{}_{\rho 3}\Gamma^\rho{}_{22} - \Gamma^3{}_{\rho 2}\Gamma^\rho{}_{23}$$

$$= \Gamma^3{}_{22,3} - \Gamma^3{}_{23,2} + \Gamma^3{}_{03}\Gamma^0{}_{22} - \Gamma^3{}_{02}\Gamma^0{}_{23} + \Gamma^3{}_{13}\Gamma^1{}_{22} - \Gamma^3{}_{12}\Gamma^1{}_{23}$$

$$+\Gamma^3{}_{23}\Gamma^2{}_{22} - \Gamma^3{}_{22}\Gamma^2{}_{23} + \Gamma^3{}_{33}\Gamma^3{}_{22} - \Gamma^3{}_{32}\Gamma^3{}_{23}$$

$$= -\frac{d}{d\theta}(cot\theta) + \left(\frac{1}{r}\right)\left(-\frac{r}{g^2}\right) - (cot\theta)(cot\theta)$$

$$= -(-cosec^2\theta) - \left(\frac{1}{g^2}\right) - cot^2\theta = 1 + cot^2\theta - \left(\frac{1}{g^2}\right) - cot^2\theta$$

$$\Rightarrow R^3{}_{232} = 1 - \left(\frac{1}{g^2}\right)$$

Therefore, also using the identity in (96), we have:

$$R^\alpha{}_{\beta\mu\nu} = g^{\alpha\rho}R_{\rho\beta\mu\nu} = g^{\alpha\rho}g_{\eta\gamma}R^{\ \eta}{}_{\beta\mu\nu}$$

$$R^3{}_{232} = g^{33}R_{3232} = g^{33}g_{22}R^{\ 2}{}_{323}$$

$$= \left(\frac{1}{r^2 sin^2\theta}\right)r^2\left(sin^2\theta\left(1 - \frac{1}{g^2}\right)\right)$$

$$\Rightarrow R^3{}_{232} = \left(1 - \left(\frac{1}{g^2}\right)\right)$$

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu}$$

$$R^0{}_{202} = \Gamma^0{}_{22,0} - \Gamma^0{}_{20,2} + \Gamma^0{}_{\rho 0}\Gamma^\rho{}_{22} - \Gamma^0{}_{\rho 2}\Gamma^\rho{}_{20}$$

$$= \Gamma^0{}_{22,0} - \Gamma^0{}_{20,2} + \Gamma^0{}_{00}\Gamma^0{}_{22} - \Gamma^0{}_{02}\Gamma^0{}_{20} + \Gamma^0{}_{10}\Gamma^1{}_{22} - \Gamma^0{}_{12}\Gamma^1{}_{20}$$

$$+\Gamma^0{}_{20}\Gamma^2{}_{22} - \Gamma^0{}_{22}\Gamma^2{}_{20} + \Gamma^0{}_{30}\Gamma^3{}_{22} - \Gamma^0{}_{32}\Gamma^3{}_{20}$$

$$= \left(\frac{f'}{f}\right)\left(-\frac{r}{g^2}\right) = -\frac{f'r}{fg^2}$$

$$\Rightarrow R^0{}_{202} = -\left(\frac{f'r}{fg^2}\right)$$

Therefore, using the identity in (96), we have:

$$R^\alpha{}_{\beta\mu\nu} = g^{\alpha\rho}R_{\rho\beta\mu\nu} = g^{\alpha\rho}g_{\eta\gamma}R^{\ \eta}{}_{\beta\mu\nu}$$

$$R^2{}_{020} = g^{22}R_{2020} = g^{22}g_{00}R^{\ 0}{}_{202}$$

$$= \left(\frac{1}{r^2}\right)(-f^2)\left(-\left(\frac{f'r}{fg^2}\right)\right)$$

$$\Rightarrow R^2{}_{020} = \frac{ff'}{rg^2}$$

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} - \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} + \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu}$$

$$R^0{}_{303} = \Gamma^0{}_{33,0} - \Gamma^0{}_{30,3} + \Gamma^0{}_{\rho 0}\Gamma^\rho{}_{33} - \Gamma^0{}_{\rho 3}\Gamma^\rho{}_{30}$$

$$= \Gamma^0{}_{33,0} - \Gamma^0{}_{30,3} + \Gamma^0{}_{00}\Gamma^0{}_{33} - \Gamma^0{}_{03}\Gamma^0{}_{30} + \Gamma^0{}_{10}\Gamma^1{}_{33} - \Gamma^0{}_{13}\Gamma^1{}_{30}$$

$$+\Gamma^0{}_{20}\Gamma^2{}_{33} - \Gamma^0{}_{23}\Gamma^2{}_{30} + \Gamma^0{}_{30}\Gamma^3{}_{33} - \Gamma^0{}_{33}\Gamma^3{}_{30}$$

$$= \left(\frac{f'}{f}\right)\left(-\frac{rsin^2\theta}{g^2}\right) = -\left(\frac{f'}{f}\right)\left(\frac{rsin^2\theta}{g^2}\right)$$

$$\Rightarrow R^0{}_{303} = -\frac{f'rsin^2\theta}{fg^2}$$

Therefore, using the identity in (96), we have:

$$R^\alpha{}_{\beta\mu\nu} = g^{\alpha\rho}R_{\rho\beta\mu\nu} = g^{\alpha\rho}g_{\eta\gamma}R^{\ \eta}{}_{\beta\mu\nu}$$

$$R^3{}_{030} = g^{33} R_{3030} = g^{33} g_{00} R^{\ 0}{}_{303}$$

$$= \left(\frac{1}{r^2 sin^2\theta}\right)(-f^2)\left(-\frac{f' r sin^2\theta}{f g^2}\right)$$

$$\Rightarrow R^3{}_{030} = \frac{f f'}{r g^2}$$

The results are summarized in the table 3.3.

Step 4:

We proceed to compute the Ricci Tensors

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \tag{97}$$

$$\Rightarrow R_{00} = R^0{}_{000} + R^1{}_{010} + R^2{}_{020} + R^3{}_{030}$$

$$= \frac{g f f'' - g' f f'}{g^3} + \frac{f f'}{r g^2} + \frac{f f'}{r g^2} = \frac{g f f'' - g' f f'}{g^3} + 2\frac{f f'}{r g^2}$$

$$\Rightarrow R_{00} = \frac{r g f f'' - r g' f f' + 2 g f f'}{r g^3}$$

$$\Rightarrow R_{01} = R^0{}_{001} + R^1{}_{011} + R^2{}_{021} + R^3{}_{031} = 0$$

$$\Rightarrow R_{02} = R^0{}_{002} + R^1{}_{012} + R^2{}_{022} + R^3{}_{032} = 0$$

$$\Rightarrow R_{03} = R^0{}_{003} + R^1{}_{013} + R^2{}_{023} + R^3{}_{033} = 0$$

$$\Rightarrow R_{11} = R^0{}_{101} + R^1{}_{111} + R^2{}_{121} + R^3{}_{131}$$

$$= \frac{g'ff' - gff''}{gf^2} + \frac{g'}{rg} + \frac{g'}{rg} = \frac{g'ff' - gff''}{gf^2} + 2\frac{g'}{rg}$$

$$\Rightarrow R_{11} = \frac{rg'ff' - rgff'' + 2f^2g'}{rgf^2}$$

Similarly, we find:

$$\Rightarrow R_{10} = R_{12} = R_{13} = 0$$

And the nonvanishing components:

$$\Rightarrow R_{22} = R^0{}_{202} + R^1{}_{212} + R^2{}_{222} + R^3{}_{232}$$

$$= -\left(\frac{f'r}{fg^2}\right) + \frac{rg'}{g^3} + \left(1 - \left(\frac{1}{g^2}\right)\right)$$

$$\Rightarrow R_{22} = \frac{-grf' + rg'f + fg^3 - fg}{fg^3}$$

$$\Rightarrow R_{20} = R_{21} = R_{23} = 0$$

$$\Rightarrow R_{33} = R^0{}_{303} + R^1{}_{313} + R^2{}_{323} + R^3{}_{333}$$

$$= -\frac{f'rsin^2\theta}{fg^2} + \frac{rg'sin^2\theta}{g^3} + sin^2\theta\left(1 - \frac{1}{g^2}\right)$$

$$\Rightarrow R_{33} = \frac{-gf'rsin^2\theta + rg'fsin^2\theta + fg^3sin^2\theta - fgsin^2\theta}{fg^3}$$

$$\Rightarrow R_{30} = R_{31} = R_{32} = 0$$

The results are summarized in the table 3.4.

Step 5:

We now solve the Einstein equation. For vacuum solutions, the scalar curvature must vanish and the Einstein equation is equivalent vanishing Ricci tensor. Equating each nonvanishing component in table 3,4 above to zero, and cancelling overall common factors, we are left with

$$rgf'' - rg'f' + 2gf' = 0 \tag{98}$$

$$R_{11} = \frac{rg'ff' - rgff'' + 2f^2g'}{rgf^2} = 0$$

$$\Rightarrow rg'ff' - rgff'' + 2f^2g' = 0$$

$$dividing\ both\ sides\ by\ f \Rightarrow rgf'' - rg'f' - 2fg' = 0 \tag{99}$$

$$R_{22} = \frac{-grf' + rg'f + fg^3 - fg}{fg^3} = 0$$

$$\Rightarrow -grf' + rg'f + fg^3 - fg = 0 \tag{100}$$

$$R_{33} = \frac{-gf'rsin^2\theta + rg'fsin^2\theta + fg^3sin^2\theta - fgsin^2\theta}{fg^3} = 0$$

$$\Rightarrow -gf'rsin^2\theta + rg'fsin^2\theta + fg^3sin^2\theta - fgsin^2\theta = 0$$

$$dividing\ both\ sides\ by\ sin^2\theta \Rightarrow -gf'r + rg'f + fg^3 - fg = 0 \tag{101}$$

Equations (100) and (101) identical and this reduces the equations to three, two of which are linearly dependent. Hence they can be easily solved as follows:

$$rgff'' - rg'ff' + 2gff' = 0$$

$$rg'ff' - rgff'' + 2f^2g' = 0$$

Dividing both sides of both equations by $\left(\frac{1}{rgf^2}\right)$, we get:

$$\left(\frac{1}{rgf^2}\right)(rgff'' - rg'ff' + 2gff') = \frac{f''}{f} - \frac{g'f'}{gf} + \frac{2}{r}\frac{f'}{f} = 0 \qquad (102)$$

and

$$\left(\frac{1}{rgf^2}\right)(rg'ff' - rgff'' + 2f^2g') = -\frac{f''}{f} + \frac{g'f'}{gf} + \frac{2}{r}\frac{g'}{g} = 0 \qquad (103)$$

Adding (102) and (103) together

$$\Rightarrow \frac{f'}{f} + \frac{g'}{g} = 0 \Rightarrow \frac{f'}{f} = -\frac{g'}{g}$$

$$\Rightarrow \frac{df}{dr}.\frac{1}{f} = -\frac{1}{g}.\frac{dg}{dr} \;\Rightarrow \frac{df}{f} = -\frac{dg}{g}$$

Therefore, integrating both sides we get:

$$\int \frac{df}{f} = \int -\frac{dg}{g} \;\Rightarrow \ln f + \ln c = -\ln g \;\; or \ln(fgc) = 0$$

$$\Rightarrow fgc = 1 \; or \; g = \frac{1}{cf} \qquad (104)$$

Now putting (104) into (101), we get:

$$-gf'r + rg'f + fg^3 - fg = 0$$

$$\Rightarrow -\left(\frac{1}{cf}\right)f'r + rf\frac{d}{dr}\left(\frac{1}{cf}\right) + f\left(\frac{1}{cf}\right)^3 - f\left(\frac{1}{cf}\right) = 0$$

$$\Rightarrow -\left(\frac{1}{cf}\right)f'r + rf\left(-\frac{1}{cf^2}\right)f' + f\left(\frac{1}{cf}\right)^3 - f\left(\frac{1}{cf}\right) = 0$$

$$\Rightarrow -\left(\frac{1}{cf}\right)f'r - \left(\frac{1}{cf}\right)f'r + \left(\frac{1}{c^3f^2}\right) - \frac{1}{c} = 0$$

$$\Rightarrow -2\left(\frac{1}{cf}\right)f'r + \left(\frac{1}{c^3f^2}\right) - \frac{1}{c} = 0$$

Multiplying through by c leads to:

$$-2\left(\frac{1}{f}\right)f'r + \left(\frac{1}{c^2f^2}\right) - 1 = 0$$

Dividing through both sides by -2r also leads to:

$$\left(\frac{f'}{f}\right) - \left(\frac{1}{2rc^2f^2}\right) + \frac{1}{2r} = 0$$

$$\Rightarrow \frac{f'}{f} = \frac{1}{2r}\left(\frac{1}{c^2f^2} - 1\right) = \frac{1}{2r}\left(\frac{1 - c^2f^2}{c^2f^2}\right)$$

Separating variables lead to:

$$\Rightarrow \frac{1}{2r} = \frac{1}{f}\frac{df}{dr}\left(\frac{c^2f^2}{1 - c^2f^2}\right)$$

Multiplying by dr and integrating,

$$\int \frac{1}{2r}dr = \int \frac{1}{f}\left(\frac{c^2f^2}{1 - c^2f^2}\right)df$$

$$\Rightarrow \frac{1}{2}(\ln r + \ln c') = \int \left(\frac{c^2 f}{1 - c^2 f^2}\right) df = -\frac{1}{2}\ln(1 - c^2 f^2)$$

$$\Rightarrow \frac{1}{2}(\ln r + \ln c') + \frac{1}{2}\ln(1 - c^2 f^2) = 0$$

$$\Rightarrow \ln rc' (1 - c^2 f^2) = 0 \Rightarrow rc'(1 - c^2 f^2) = 1$$

$$\Rightarrow (1 - c^2 f^2) = \frac{1}{rc'}$$

$$\Rightarrow f = \frac{1}{c}\sqrt{1 - \frac{1}{rc'}} \qquad (105)$$

Therefore, putting (105) into (101) we get:

$$g = \frac{1}{cf} = \frac{1}{c\left(\frac{1}{c}\sqrt{1 - \frac{1}{rc'}}\right)} = \frac{1}{\sqrt{1 - \frac{1}{rc'}}} \qquad (106)$$

Now recall the form of the metric,

$$ds^2 = -f^2 dt^2 + g^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \qquad (107)$$

In form of solid angle this becomes:

$$ds^2 = -f^2 dt^2 + g^2 dr^2 + r^2 d\Omega^2 \qquad (108)$$

Substituting (106) and (108),

$$ds^2 = -\left(\frac{1}{c}\sqrt{1 - \frac{1}{rc'}}\right)^2 dt^2 + \left(\frac{1}{\sqrt{1 - \frac{1}{rc'}}}\right)^2 dr^2 + r^2 d\Omega^2$$

$$= -\frac{1}{c^2}\left(1-\frac{1}{rc'}\right)dt^2 + \left(\frac{1}{1-\frac{1}{rc'}}\right)dr^2 + r^2 d\Omega^2 \tag{109}$$

Step 6:

Next, we determine the remaining constant, c'. To do this we require the geodesic equation, which follows by finding the extremum of the arclength.

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

$$\Rightarrow s = \int ds = \int_{c(\lambda)} \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda$$

Varying.

$$\delta s = 0 = \delta \int ds = \delta \int_{c(\lambda)} \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda$$

$$= \int_{c(\lambda)} \left\{\frac{1}{2}\frac{1}{\sqrt{g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}\left(g_{\mu\nu,\rho}\delta x^\rho \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d\delta x^\nu}{d\lambda}\right)\right\} d\lambda$$

$$\int_{c(\lambda)} \left\{\frac{1}{2}\frac{1}{\sqrt{g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}\left(g_{\mu\nu,\rho}\delta x^\rho \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + 2g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)\right\} d\lambda$$

But

$$\frac{d}{d\lambda}\left[\frac{g_{\mu\nu}\frac{dx^\nu}{d\lambda}\delta x^\mu}{\sqrt{g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}\right] = \frac{g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}} + \delta x^\rho \frac{d}{d\lambda}\left[\frac{g_{\rho\nu}\frac{dx^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}\right]$$

so

$$0 = \int_{c(\lambda)} \left\{ \frac{1}{2} \left[\frac{g_{\mu\nu,\rho} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \delta x^\rho}{\sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}} \right] + \frac{d}{d\lambda} \left[\frac{g_{\mu\nu} \frac{dx^\nu}{d\lambda} \delta x^\mu}{\sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}} \right] - \delta x^\rho \frac{d}{d\lambda} \left[\frac{g_{\rho\nu} \frac{dx^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}} \right] \right\} d\lambda$$

The middle term above vanishes because it integrates to the endpoints and the variation vanishes at the endpoints.

$$\Rightarrow \int_{c(\lambda)} \frac{1}{2} \left[\frac{g_{\mu\nu,\rho} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \delta x^\rho}{\sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}} \right] - \delta x^\rho \frac{d}{d\lambda} \left[\frac{g_{\rho\nu} \frac{dx^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}} \right] = 0$$

$$\Rightarrow \int_{c(\lambda)} \left\{ \frac{1}{2} \left[\frac{g_{\mu\nu,\rho} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}} \right] - \frac{d}{d\lambda} \left[\frac{g_{\rho\nu} \frac{dx^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}} \right] \right\} \delta x^\rho = 0$$

$$\Rightarrow \frac{1}{2} \left[\frac{g_{\mu\nu,\rho} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}} \right] - \frac{d}{d\lambda} \left[\frac{g_{\rho\nu} \frac{dx^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}} \right] = 0$$

$$\Rightarrow \frac{1}{2} \left[\frac{g_{\mu\nu,\rho} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}} \right] - \frac{d}{d\lambda} \left[\frac{g_{\rho\nu} \frac{dx^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}} \right] = 0$$

Now choose $\lambda = s$ so that:

$$g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = \frac{ds^2}{d\lambda d\lambda} = \frac{ds^2}{ds^2} = 1$$

and the equation reduces to:

$$\Rightarrow \frac{1}{2}\left[\frac{g_{\mu\nu,\rho}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}\right] - \frac{d}{d\lambda}\left[\frac{g_{\rho\nu}\frac{dx^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}\right] = \frac{1}{2}\left[g_{\mu\nu,\rho}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right] - \frac{d}{ds}\left[g_{\rho\nu}\frac{dx^\nu}{ds}\right] = 0$$

$$\Rightarrow \frac{1}{2}\left[g_{\mu\nu,\rho}u^\mu u^\nu\right] - \frac{d}{ds}\left[g_{\rho\nu}u^\nu\right] = 0$$

$$\Rightarrow \frac{1}{2}\left[g_{\mu\nu,\rho}u^\mu u^\nu\right] - g_{\rho\nu,\mu}\frac{dx^\mu}{ds}u^\nu - g_{\rho\nu}\frac{du^\nu}{ds} = 0 \qquad (110)$$

where we have used,

$$\frac{d}{ds}\left[g_{\rho\nu}u^\nu\right] = \frac{d}{ds}\left[g_{\rho\nu}\big(x(s)\big)u^\nu(s)\right] = \frac{d}{ds}\left[g_{\rho\nu}\big(x(s)\big)\right].u^\nu(s) + g_{\rho\nu}\big(x(s)\big).\frac{du^\nu(s)}{ds}$$

$$= \frac{\partial g_{\rho\nu}}{dx^\mu}.\frac{dx^\mu}{ds}.u^\nu + g_{\rho\nu}\frac{du^\nu}{ds}$$

Therefore, continuing from (110), we get:

$$\Rightarrow \frac{1}{2}\left[g_{\mu\nu,\rho}u^\mu u^\nu\right] - g_{\rho\nu,\mu}\frac{dx^\mu}{ds}u^\nu - g_{\rho\nu}\frac{du^\nu}{ds} = 0$$

$$\Rightarrow g_{\rho\nu}\frac{du^\nu}{ds} = \frac{1}{2}\left[g_{\mu\nu,\rho}u^\mu u^\nu\right] - g_{\rho\nu,\mu}\frac{dx^\mu}{ds}u^\nu = \frac{1}{2}\left[g_{\mu\nu,\rho}u^\mu u^\nu\right] - g_{\rho\nu,\mu}u^\mu u^\nu$$

For the last term, we symmetrize:

$$g_{\rho\nu,\mu}u^\mu u^\nu = g_{\rho\mu,\nu}u^\nu u^\mu \Rightarrow g_{\rho\nu,\mu}u^\mu u^\nu = \frac{1}{2}\left(g_{\rho\nu,\mu}u^\mu u^\nu + g_{\rho\mu,\nu}u^\nu u^\mu\right)$$

So that

$$g_{\rho\nu}\frac{du^\nu}{ds} = \frac{1}{2}\left[g_{\mu\nu,\rho}u^\mu u^\nu\right] - g_{\rho\nu,\mu}u^\mu u^\nu = \frac{1}{2}\left[g_{\mu\nu,\rho}u^\mu u^\nu\right] - \frac{1}{2}\left(g_{\rho\nu,\mu}u^\mu u^\nu + g_{\rho\mu,\nu}u^\nu u^\mu\right)$$

$$= \frac{1}{2}\left(g_{\mu\nu,\rho} - g_{\rho\nu,\mu} - g_{\rho\mu,\nu}\right)u^{\mu}u^{\nu} = -\Gamma_{\rho\mu\nu}u^{\mu}u^{\nu}$$

$$\Rightarrow g_{\rho\nu}\frac{du^{\nu}}{ds} = -\Gamma_{\rho\mu\nu}u^{\mu}u^{\nu}$$

Therefore, multiplying both sides by $g^{\alpha\rho}$, we get the geodesic equation;

$$g^{\alpha\rho}\left(g_{\rho\nu}\frac{du^{\nu}}{ds}\right) = g^{\alpha\rho}\left(-\Gamma_{\rho\mu\nu}u^{\mu}u^{\nu}\right) = -\Gamma^{\alpha}{}_{\mu\nu}u^{\mu}u^{\nu}$$

since

$$\frac{du^{\alpha}}{ds} = -\Gamma^{\alpha}{}_{\mu\nu}u^{\mu}u^{\nu}$$

we may write the geodesic equation as:

$$\Rightarrow \frac{du^{\alpha}}{ds} + \Gamma^{\alpha}{}_{\mu\nu}u^{\mu}u^{\nu} = 0 = u^{\nu}D_{\nu}u^{\alpha} \tag{111}$$

Now we apply the geodesic equation to a circular obit at large value of r, with

$$D_{\nu}u^{\alpha} = \partial_{\nu}u^{\alpha} + u^{\mu}\Gamma^{\alpha}{}_{\mu\nu} \Rightarrow u^{\nu}D_{\nu}u^{\alpha} = \frac{dx^{\nu}}{ds}\frac{du^{\alpha}}{dx^{\nu}} + \Gamma^{\alpha}{}_{\mu\nu}u^{\mu}u^{\nu}$$

$$\therefore u^{\nu}D_{\nu}u^{\alpha} = \frac{du^{\alpha}}{ds} + \Gamma^{\alpha}{}_{\mu\nu}u^{\mu}u^{\nu}$$

From (24), we have:

$$\frac{du^{\alpha}}{ds} = -\Gamma^{\alpha}{}_{\mu\nu}u^{\mu}u^{\nu}$$

$$\Rightarrow \frac{du^{\alpha}}{d\tau} = -\Gamma^{\alpha}{}_{\mu\nu}u^{\mu}u^{\nu}$$

where

$$u^{\mu} = \gamma(c, \boldsymbol{v}) \approx (c, \boldsymbol{v})$$

then choosing $c = 1$, we get $g = \frac{1}{cf} = \frac{1}{f}$

$$\therefore \Gamma^{1}_{00} = \frac{ff'}{g^2} = f^3 f' = \left(\sqrt{1 - \frac{1}{rc'}}\right)^{\frac{3}{2}} \left[\frac{1}{2}\left(\frac{1}{c'r^2}\right)\left(\frac{1}{\sqrt{1 - \frac{1}{rc'}}}\right)\right]$$

Γ^{1}_{00} is the term that contributes most to our central force problem

$$\Gamma^{1}_{00} \approx \frac{1}{2c'r^2} \; assuming \; 1 - \frac{1}{rc'} \approx 1 \; for \; large \; r$$

$$a_i = \frac{dv^i}{dt} = -\Gamma^{i}_{00} c^2 - \Gamma^{i}_{0j} c^2 \frac{v^i}{c} - \cdots$$

$$\therefore a_r = \frac{dv^r}{dt} = -\Gamma^{1}_{00} c^2 - \Gamma^{1}_{0j} c^2 \frac{v^i}{c} - \cdots \approx -\frac{1}{2c'r^2} = -\frac{GM}{r^2}$$

$$\Rightarrow c' = \frac{1}{2GM}$$

$$\Rightarrow f = \sqrt{1 - \frac{1}{rc'}} = \sqrt{1 - \frac{2GM}{r}}$$

$$\Rightarrow ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2 d\Omega^2 \qquad (112)$$

APPENDIX II

DETAILED PROOF OF BIRKHOFF THEOREM IN WEYL GRAVITY

We have chosen the general time-dependent spherically symmetric line-element of the form given in (53), and (54). We rewrite them here again as follows

$$ds^2 = -f^2(r,t)dt^2 + g^2(r,t)dr^2 + r^2(r)d\theta + r^2 sin^2\theta(r)d\varphi^2 \qquad (113)$$

$$g_{ab} = \begin{bmatrix} -f^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{bmatrix} \qquad (114)$$

$$g^{a\,b} = \begin{bmatrix} -f^{-2} & 0 & 0 & 0 \\ 0 & g^{-2} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2} cosec^2\theta \end{bmatrix} \qquad (115)$$

We aim at finding the Schwarzschild Solutions from the basics by following the steps below:

Step 1:

We first we find the non-zero components of the Affine connection or Christoffel symbol.

$$\Gamma_{\alpha\mu\nu} = \frac{1}{2}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right) \qquad (116)$$

Owing to the fact that we already know that,

$$\Gamma_{\alpha\mu\nu} = \Gamma_{\alpha\nu\mu} \text{ and } \Gamma_{\alpha\mu\nu} = -\Gamma_{\mu\alpha\nu} \qquad (117)$$

Working out (116) with the help of (117) we get thirteen non-zero components for the affine connection.

$$\Gamma_{000} = \frac{1}{2}\left(g_{00,0} + g_{00,0} - g_{00,0}\right) = \frac{(-\dot{f^2})}{2} = \frac{-2f\dot{f}}{2} = -f\dot{f}$$

$$\Gamma_{110} = \frac{1}{2}\left(g_{11,0} + g_{10,1} - g_{10,1}\right) = \frac{(\dot{g^2})}{2} = \frac{2g\dot{g}}{2} = g\dot{g}$$

$$\Gamma_{110} = \Gamma_{101} = g\dot{g} \ and \ \Gamma_{011} = -\Gamma_{101} = -g\dot{g}$$

$$\Gamma_{001} = \frac{1}{2}\left(g_{00,1} + g_{01,0} - g_{01,0}\right) = \frac{(-f^2)'}{2} = -\frac{2ff'}{2} = -ff'$$

$$\Gamma_{010} = \Gamma_{001} = -ff' and \ \Gamma_{100} = -\Gamma_{010} = ff'$$

$$\Gamma_{111} = \frac{1}{2}\left(g_{11,1} + g_{11,1} - g_{11,1}\right) = \frac{(g^2)'}{2} = \frac{2gg'}{2} = gg'$$

$$\Gamma_{221} = \frac{1}{2}\left(g_{22,1} + g_{21,2} - g_{21,2}\right) = \frac{(r^2)'}{2} = \frac{2r}{2} = r$$

$$\Gamma_{221} = \Gamma_{212} = r \ \ and \ \Gamma_{122} = -\Gamma_{212} = -r$$

$$\Gamma_{331} = \frac{1}{2}\left(g_{33,1} + g_{31,3} - g_{31,3}\right) = \frac{d}{dr}\frac{(r^2 sin^2\theta)}{2} = \frac{2r sin^2\theta}{2} = r sin^2\theta$$

$$\Gamma_{313} = \Gamma_{331} = r sin^2\theta \ \ and \ \ \Gamma_{133} = -\Gamma_{313} = -r sin^2\theta$$

$$\Gamma_{332} = \frac{1}{2}\left(g_{33,2} + g_{32,3} - g_{32,3}\right) = \frac{d}{d\theta}\frac{(r^2 sin^2\theta)}{2} = \frac{2r^2 sin\theta cos\theta}{2} = r^2 sin\theta cos\theta$$

$$\Gamma_{323} = \Gamma_{332} = r^2 sin\theta cos\theta \ \ \ and \ \ \Gamma_{233} = -\Gamma_{323} = -r^2 sin\theta cos\theta$$

The results are summarized in the table 3.5.

Step 2:

We now compute the corresponding mixed form of the Affine connection

$$\Gamma^{\alpha}{}_{\mu\nu} = g^{\alpha\beta}\Gamma_{\beta\mu\nu} \tag{118}$$

We also make good use of the identities that,

$$\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu} \tag{119}$$

Due to the symmetry in our metric, it becomes easy to do this and the only surviving terms are:

$$\Gamma^{0}{}_{01} = g^{00}\Gamma_{001} = \left(-\frac{1}{f^2}\right)(-ff') = \frac{f'}{f} = \Gamma^{0}{}_{10}$$

$$\Gamma^{0}{}_{00} = g^{00}\Gamma_{000} = \left(-\frac{1}{f^2}\right)(-f\dot{f}) = \frac{\dot{f}}{f}$$

$$\Gamma^{1}{}_{00} = g^{11}\Gamma_{100} = \frac{1}{g^2}(ff') = \frac{ff'}{g^2}$$

$$\Gamma^{1}{}_{10} = g^{11}\Gamma_{110} = \left(\frac{1}{g^2}\right)(g\dot{g}) = \frac{\dot{g}}{g} = \Gamma^{1}{}_{01}$$

$$\Gamma^{0}{}_{11} = g^{00}\Gamma_{011} = \left(-\frac{1}{f^2}\right)(-g\dot{g}) = \frac{g\dot{g}}{f^2}$$

$$\Gamma^{1}{}_{11} = g^{11}\Gamma_{111} = \left(\frac{1}{g^2}\right)(gg') = \frac{g'}{g}$$

$$\Gamma^{2}{}_{21} = g^{22}\Gamma_{221} = \left(\frac{1}{r^2}\right)(r) = \frac{1}{r} = \Gamma^{2}{}_{12}$$

$$\Gamma^{1}{}_{22} = g^{11}\Gamma_{122} = \left(\frac{1}{g^2}\right)(-r) = -\frac{r}{g^2}$$

$$\Gamma^3{}_{31} = g^{33}\Gamma_{331} = \left(\frac{1}{r^2 sin^2\theta}\right)(rsin^2\theta) = \frac{1}{r} = \Gamma^3{}_{13}$$

$$\Gamma^1{}_{33} = g^{11}\Gamma_{133} = \left(\frac{1}{g^2}\right)(-rsin^2\theta) = -\frac{rsin^2\theta}{g^2}$$

$$\Gamma^3{}_{32} = g^{33}\Gamma_{332} = \left(\frac{1}{r^2 sin^2\theta}\right)(r^2 sin\theta cos\theta) = cot = \Gamma^3{}_{23}$$

$$\Gamma^2{}_{33} = g^{22}\Gamma_{233} = \left(\frac{1}{r^2}\right)(-r^2 sin\theta cos\theta) = -sin\theta cos\theta$$

The results are summarized in the table 3.6.

Step 3:

We proceed to compute the curvature.

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu} \tag{120}$$

$$R^1{}_{010} = \Gamma^1{}_{00,1} - \Gamma^1{}_{01,0} + \Gamma^1{}_{\rho 1}\Gamma^\rho{}_{00} - \Gamma^1{}_{\rho 0}\Gamma^\rho{}_{01}$$

$$= \Gamma^1{}_{00,1} - \Gamma^1{}_{01,0} + \Gamma^1{}_{11}\Gamma^1{}_{00} - \Gamma^1{}_{10}\Gamma^1{}_{01} + \Gamma^1{}_{01}\Gamma^0{}_{00} - \Gamma^1{}_{00}\Gamma^0{}_{01}$$

$$= \frac{d}{dr}\left(\frac{ff'}{g^2}\right) - \frac{d}{dt}\left(\frac{\dot{g}}{g}\right) + \left(\frac{g'}{g}\right)\left(\frac{ff'}{g^2}\right) - \left(\frac{\dot{g}}{g}\right)\left(\frac{\dot{g}}{g}\right) + \left(\frac{\dot{g}}{g}\right)\left(\frac{\dot{f}}{f}\right) - \left(\frac{f'}{f}\right)\left(\frac{ff'}{g^2}\right)$$

$$= \frac{d}{dr}\left(\frac{ff'}{g^2}\right) - \frac{d}{dt}\left(\frac{\dot{g}}{g}\right) + \left(\frac{g'ff'}{g^3}\right) - \left(\frac{\dot{g}}{g}\right)^2 + \frac{\dot{g}\dot{f}}{gf} - \left(\frac{f'}{g}\right)^2$$

$$= \frac{ff''}{g^2} + \left(\frac{f'}{g}\right)^2 + ff'\left(-\frac{2g'}{g^3}\right) - \frac{\ddot{g}}{g} + \frac{\dot{g}\dot{g}}{g^2} + \frac{g'ff'}{g^3} - \left(\frac{\dot{g}}{g}\right)^2 + \frac{\dot{g}\dot{f}}{gf} - \left(\frac{f'}{g}\right)^2$$

$$= \frac{ff''}{g^2} - \frac{g'ff'}{g^3} - \frac{\ddot{g}}{g} + \frac{\dot{g}\dot{f}}{gf} = \frac{gf^2f'' - g'f^2f' - g^2f\ddot{g} + g^2\dot{g}\dot{f}}{g^3f}$$

$$\Rightarrow R^1{}_{010} = \frac{gf^2f'' - g'f^2f' - g^2f\ddot{g} + g^2\dot{g}\dot{f}}{g^3f}$$

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu}$$

$$R^0{}_{101} = \Gamma^0{}_{11,0} - \Gamma^0{}_{10,1} + \Gamma^0{}_{\rho 0}\Gamma^\rho{}_{11} - \Gamma^0{}_{\rho 1}\Gamma^\rho{}_{10}$$

$$= \Gamma^0{}_{11,0} - \Gamma^0{}_{10,1} + \Gamma^0{}_{00}\Gamma^0{}_{11} - \Gamma^0{}_{01}\Gamma^0{}_{10} + \Gamma^0{}_{10}\Gamma^1{}_{11} - \Gamma^0{}_{11}\Gamma^1{}_{10}$$

$$= \frac{d}{dt}\left(\frac{g\dot{g}}{f^2}\right) - \frac{d}{dr}\left(\frac{f'}{f}\right) + \left(\frac{\dot{f}}{f}\right)\left(\frac{g\dot{g}}{f^2}\right) - \left(\frac{f'}{f}\right)^2 + \left(\frac{g'}{g}\right)\left(\frac{f'}{f}\right) - \left(\frac{g\dot{g}}{f^2}\right)\left(\frac{\dot{g}}{g}\right)$$

$$= \left[\left(\frac{\dot{g}\dot{g}}{f^2}\right) + \left(\frac{g\ddot{g}}{f^2}\right) - 2\left(\frac{g\dot{g}\dot{f}}{f^3}\right)\right] - \left(\frac{ff'' - f'f'}{f^2}\right) + \left(\frac{g\dot{g}\dot{f}}{f^3}\right) - \left(\frac{f'}{f}\right)^2 + \left(\frac{g'f'}{gf}\right) - \left(\frac{\dot{g}}{g}\right)^2$$

$$= \left(\frac{g\ddot{g}}{f^2}\right) - \left(\frac{g\dot{g}\dot{f}}{f^3}\right) - \frac{f''}{f} + \left(\frac{f'}{f}\right)^2 - \left(\frac{f'}{f}\right)^2 + \left(\frac{g'f'}{gf}\right)$$

$$\Rightarrow R^0{}_{101} = \left(\frac{g\ddot{g}}{f^2}\right) - \left(\frac{g\dot{g}\dot{f}}{f^3}\right) - \frac{f''}{f} + \left(\frac{g'f'}{gf}\right) = \frac{g^2f\ddot{g} - g^2\dot{g}\dot{f} - gf^2f'' + g'f^2f'}{gf^3}$$

Therefore, using the identity;

$$R^\alpha{}_{\beta\mu\nu} = g^{\alpha\rho}R_{\rho\beta\mu\nu} = g^{\alpha\rho}g_{\eta\gamma}R^{\ \eta}{}_{\beta\mu\nu} \tag{121}$$

$$R^1{}_{010} = g^{11}R_{1010} = g^{11}g_{00}R^0{}_{101} = -\frac{f^2}{g^2}\left[\frac{g^2f\ddot{g} - g^2\dot{g}\dot{f} - gf^2f'' + g'f^2f'}{gf^3}\right]$$

$$\Rightarrow R^1{}_{010} = \frac{gf^2f'' - g'f^2f' - g^2f\ddot{g} + g^2\dot{g}\dot{f}}{g^3f}$$

which confirms what we got before for $R^1{}_{010}$, and confirms the validity of the identity (121).

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu}$$

$$R^1{}_{212} = \Gamma^1{}_{22,1} - \Gamma^1{}_{21,2} + \Gamma^1{}_{\rho 1}\Gamma^\rho{}_{22} - \Gamma^1{}_{\rho 2}\Gamma^\rho{}_{21}$$

$$= \Gamma^1{}_{22,1} - \Gamma^1{}_{21,2} + \Gamma^1{}_{11}\Gamma^1{}_{22} - \Gamma^1{}_{12}\Gamma^1{}_{21} + \Gamma^1{}_{01}\Gamma^0{}_{22} - \Gamma^1{}_{02}\Gamma^0{}_{21}$$

$$+\Gamma^1{}_{21}\Gamma^2{}_{22} - \Gamma^1{}_{22}\Gamma^2{}_{21} + \Gamma^1{}_{31}\Gamma^3{}_{22} - \Gamma^1{}_{32}\Gamma^3{}_{21}$$

$$= \frac{d}{dr}\left(-\frac{r}{g^2}\right) + \left(\frac{g'}{g}\right)\left(-\frac{r}{g^2}\right) - \left(\frac{1}{r}\right)\left(-\frac{r}{g^2}\right) = -\left(\frac{g^2.1 - r.2gg'}{g^4}\right) - \left(\frac{g'r}{g^3}\right) + \left(\frac{1}{g^2}\right)$$

$$R^1{}_{212} = \frac{-g^2 + 2rgg' - gg'r + g^2}{g^4} = \frac{rgg'}{g^4}$$

$$\Rightarrow R^1{}_{212} = \frac{rg'}{g^3}$$

Therefore, using the identity in (121), we have:

$$R^\alpha{}_{\beta\mu\nu} = g^{\alpha\rho}R_{\rho\beta\mu\nu} = g^{\alpha\rho}g_{\eta\gamma}R^{\,\eta}{}_{\beta\mu\nu}$$

$$R^2{}_{121} = g^{22}R_{2121} = g^{22}g_{11}R^{\,1}{}_{212}$$

$$= \left(\frac{1}{r^2}\right)(g^2)\left(\frac{rg'}{g^3}\right) = \frac{g'}{rg}$$

$$\Rightarrow R^2{}_{121} = \frac{g'}{rg}$$

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu}$$

$$R^{1}{}_{313} = \Gamma^{1}{}_{33,1} - \Gamma^{1}{}_{31,3} + \Gamma^{1}{}_{\rho 1}\Gamma^{\rho}{}_{33} - \Gamma^{1}{}_{\rho 3}\Gamma^{\rho}{}_{31}$$

$$= \Gamma^{1}{}_{33,1} - \Gamma^{1}{}_{31,3} + \Gamma^{1}{}_{11}\Gamma^{1}{}_{33} - \Gamma^{1}{}_{13}\Gamma^{1}{}_{31} + \Gamma^{1}{}_{01}\Gamma^{0}{}_{33} - \Gamma^{1}{}_{03}\Gamma^{0}{}_{31}$$

$$+\Gamma^{1}{}_{21}\Gamma^{2}{}_{33} - \Gamma^{1}{}_{23}\Gamma^{2}{}_{31} + \Gamma^{1}{}_{31}\Gamma^{3}{}_{33} - \Gamma^{1}{}_{33}\Gamma^{3}{}_{31}$$

$$= \frac{d}{dr}\left(-\frac{rsin^2\theta}{g^2}\right) + \left(\frac{g'}{g}\right)\left(-\frac{rsin^2\theta}{g^2}\right) - \left(-\frac{rsin^2\theta}{g^2}\right)\left(\frac{1}{r}\right)$$

$$= -\left(\frac{g^2sin^2\theta - rsin^2\theta.2gg'}{g^4}\right) - \left(\frac{g'rsin^2\theta}{g^3}\right) + \left(\frac{sin^2\theta}{g^2}\right)$$

$$= \left(\frac{2rgg'sin^2\theta - g^2sin^2\theta - rgg'sin^2\theta + g^2sin^2\theta}{g^4}\right) = \frac{rgg'sin^2\theta}{g^4}$$

$$\Rightarrow R^{1}{}_{313} = \left(\frac{rg'sin^2\theta}{g^3}\right)$$

Therefore, using the identity in (121), we have:

$$R^{\alpha}{}_{\beta\mu\nu} = g^{\alpha\rho}R_{\rho\beta\mu\nu} = g^{\alpha\rho}g_{\eta\gamma}R^{\ \eta}{}_{\beta\mu\nu}$$

$$R^{3}{}_{131} = g^{33}R_{3131} = g^{33}g_{11}R^{\ 1}{}_{313}$$

$$= \left(\frac{1}{r^2sin^2\theta}\right)g^2\left(\frac{rg'sin^2\theta}{g^3}\right)$$

$$\Rightarrow R^{3}{}_{131} = \frac{g'}{rg}$$

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\rho\mu}\Gamma^{\rho}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\rho\nu}\Gamma^{\rho}{}_{\beta\mu}$$

$$R^{2}{}_{323} = \Gamma^{2}{}_{33,2} - \Gamma^{2}{}_{32,3} + \Gamma^{2}{}_{\rho 2}\Gamma^{\rho}{}_{33} - \Gamma^{2}{}_{\rho 3}\Gamma^{\rho}{}_{32}$$

$$= \Gamma^2{}_{33,2} - \Gamma^2{}_{32,3} + \Gamma^2{}_{02}\Gamma^0{}_{33} - \Gamma^2{}_{03}\Gamma^0{}_{32} + \Gamma^2{}_{12}\Gamma^1{}_{33} - \Gamma^2{}_{13}\Gamma^1{}_{32}$$

$$+\Gamma^2{}_{22}\Gamma^2{}_{33} - \Gamma^2{}_{23}\Gamma^2{}_{32} + \Gamma^2{}_{32}\Gamma^3{}_{33} - \Gamma^2{}_{33}\Gamma^3{}_{32}$$

$$= \frac{d}{d\theta}(-sin\theta cos\theta) + \left(\frac{1}{r}\right)\left(-\frac{rsin^2\theta}{g^2}\right) - (-sin\theta cos\theta)(cot\theta)$$

$$= sin^2\theta - cos^2\theta - \frac{sin^2\theta}{g^2} + cos^2\theta$$

$$\Rightarrow R^2{}_{323} = sin^2\theta\left(1 - \frac{1}{g^2}\right)$$

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} - \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} + \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu}$$

$$R^3{}_{232} = \Gamma^3{}_{22,3} - \Gamma^3{}_{23,2} + \Gamma^3{}_{\rho 3}\Gamma^\rho{}_{22} - \Gamma^3{}_{\rho 2}\Gamma^\rho{}_{23}$$

$$= \Gamma^3{}_{22,3} - \Gamma^3{}_{23,2} + \Gamma^3{}_{03}\Gamma^0{}_{22} - \Gamma^3{}_{02}\Gamma^0{}_{23} + \Gamma^3{}_{13}\Gamma^1{}_{22} - \Gamma^3{}_{12}\Gamma^1{}_{23}$$

$$+\Gamma^3{}_{23}\Gamma^2{}_{22} - \Gamma^3{}_{22}\Gamma^2{}_{23} + \Gamma^3{}_{33}\Gamma^3{}_{22} - \Gamma^3{}_{32}\Gamma^3{}_{23}$$

$$= -\frac{d}{d\theta}(cot\theta) + \left(\frac{1}{r}\right)\left(-\frac{r}{g^2}\right) - (cot\theta)(cot\theta)$$

$$= -(-cosec^2\theta) - \left(\frac{1}{g^2}\right) - cot^2\theta = 1 + cot^2\theta - \left(\frac{1}{g^2}\right) - cot^2\theta$$

$$\Rightarrow R^3{}_{232} = 1 - \left(\frac{1}{g^2}\right)$$

Therefore, also using the identity in (121), we have:

$$R^\alpha{}_{\beta\mu\nu} = g^{\alpha\rho}R_{\rho\beta\mu\nu} = g^{\alpha\rho}g_{\eta\gamma}R^{\,\eta}{}_{\beta\mu\nu}$$

$$R^{3}{}_{232} = g^{33}R_{3232} = g^{33}g_{22}R^{\ 2}{}_{323}$$

$$= \left(\frac{1}{r^2 sin^2\theta}\right) r^2 \left(sin^2\theta\left(1 - \frac{1}{g^2}\right)\right)$$

$$\Rightarrow R^{3}{}_{232} = \left(1 - \left(\frac{1}{g^2}\right)\right)$$

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\rho\mu}\Gamma^{\rho}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\rho\nu}\Gamma^{\rho}{}_{\beta\mu}$$

$$R^{0}{}_{202} = \Gamma^{0}{}_{22,0} - \Gamma^{0}{}_{20,2} + \Gamma^{0}{}_{\rho 0}\Gamma^{\rho}{}_{22} - \Gamma^{0}{}_{\rho 2}\Gamma^{\rho}{}_{20}$$

$$= \Gamma^{0}{}_{22,0} - \Gamma^{0}{}_{20,2} + \Gamma^{0}{}_{00}\Gamma^{0}{}_{22} - \Gamma^{0}{}_{02}\Gamma^{0}{}_{20} + \Gamma^{0}{}_{10}\Gamma^{1}{}_{22} - \Gamma^{0}{}_{12}\Gamma^{1}{}_{20}$$

$$+\Gamma^{0}{}_{20}\Gamma^{2}{}_{22} - \Gamma^{0}{}_{22}\Gamma^{2}{}_{20} + \Gamma^{0}{}_{30}\Gamma^{3}{}_{22} - \Gamma^{0}{}_{32}\Gamma^{3}{}_{20}$$

$$= \left(\frac{f'}{f}\right)\left(-\frac{r}{g^2}\right) = -\frac{f'r}{fg^2}$$

$$\Rightarrow R^{0}{}_{202} = -\left(\frac{f'r}{fg^2}\right)$$

Therefore, using the identity in (121), we have:

$$R^{\alpha}{}_{\beta\mu\nu} = g^{\alpha\rho}R_{\rho\beta\mu\nu} = g^{\alpha\rho}g_{\eta\gamma}R^{\ \eta}{}_{\beta\mu\nu}$$

$$R^{2}{}_{020} = g^{22}R_{2020} = g^{22}g_{00}R^{\ 0}{}_{202}$$

$$= \left(\frac{1}{r^2}\right)(-f^2)\left(-\left(\frac{f'r}{fg^2}\right)\right)$$

$$\Rightarrow R^{2}{}_{020} = \frac{ff'}{rg^2}$$

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} - \Gamma^{\alpha}{}_{\rho\mu}\Gamma^{\rho}{}_{\beta\nu} + \Gamma^{\alpha}{}_{\rho\nu}\Gamma^{\rho}{}_{\beta\mu}$$

$$R^{0}{}_{303} = \Gamma^{0}{}_{33,0} - \Gamma^{0}{}_{30,3} + \Gamma^{0}{}_{\rho 0}\Gamma^{\rho}{}_{33} - \Gamma^{0}{}_{\rho 3}\Gamma^{\rho}{}_{30}$$

$$= \Gamma^{0}{}_{33,0} - \Gamma^{0}{}_{30,3} + \Gamma^{0}{}_{00}\Gamma^{0}{}_{33} - \Gamma^{0}{}_{03}\Gamma^{0}{}_{30} + \Gamma^{0}{}_{10}\Gamma^{1}{}_{33} - \Gamma^{0}{}_{13}\Gamma^{1}{}_{30}$$

$$+\Gamma^{0}{}_{20}\Gamma^{2}{}_{33} - \Gamma^{0}{}_{23}\Gamma^{2}{}_{30} + \Gamma^{0}{}_{30}\Gamma^{3}{}_{33} - \Gamma^{0}{}_{33}\Gamma^{3}{}_{30}$$

$$= \left(\frac{f'}{f}\right)\left(-\frac{rsin^2\theta}{g^2}\right) = -\left(\frac{f'}{f}\right)\left(\frac{rsin^2\theta}{g^2}\right)$$

$$\Rightarrow R^{0}{}_{303} = -\frac{f'rsin^2\theta}{fg^2}$$

Therefore, using the identity in (121), we have:

$$R^{\alpha}{}_{\beta\mu\nu} = g^{\alpha\rho}R_{\rho\beta\mu\nu} = g^{\alpha\rho}g_{\eta\gamma}R^{\ \eta}{}_{\beta\mu\nu}$$

$$R^{3}{}_{030} = g^{33}R_{3030} = g^{33}g_{00}R^{\ 0}{}_{303}$$

$$= \left(\frac{1}{r^2sin^2\theta}\right)(-f^2)\left(-\frac{f'rsin^2\theta}{fg^2}\right)$$

$$\Rightarrow R^{3}{}_{030} = \frac{ff'}{rg^2}$$

The results are summarized in the table 3.7.

Step 4:

We proceed to compute the Ricci tensors.

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$$

$$\Rightarrow R_{00} = R^0{}_{000} + R^1{}_{010} + R^2{}_{020} + R^3{}_{030}$$

$$= \frac{gf^2f'' - g'f^2f' - g^2f\ddot{g} + g^2\dot{g}\dot{f}}{g^3f} + \frac{ff'}{rg^2} + \frac{ff'}{rg^2}$$

$$= \frac{rgf^2f'' - rg'f^2f' - rg^2f\ddot{g} + rg^2\dot{g}\dot{f} + gf^2f' + gf^2f'}{g^3fr}$$

$$\Rightarrow R_{00} = \frac{rgf^2f'' - rg'f^2f' - rg^2f\ddot{g} + rg^2\dot{g}\dot{f} + 2gf^2f'}{g^3fr}$$

$$\Rightarrow R_{01} = R^0{}_{001} + R^1{}_{011} + R^2{}_{021} + R^3{}_{031} = 0$$

$$\Rightarrow R_{02} = R^0{}_{002} + R^1{}_{012} + R^2{}_{022} + R^3{}_{032} = 0$$

$$\Rightarrow R_{03} = R^0{}_{003} + R^1{}_{013} + R^2{}_{023} + R^3{}_{033} = 0$$

$$\Rightarrow R_{11} = R^0{}_{101} + R^1{}_{111} + R^2{}_{121} + R^3{}_{131}$$

$$= \frac{g^2f\ddot{g} - g^2\dot{g}\dot{f} - gf^2f'' + g'f^2f'}{gf^3} + \frac{g'}{rg} + \frac{g'}{rg}$$

$$= \frac{rg^2f\ddot{g} - rg^2\dot{g}\dot{f} - rgf^2f'' + rg'f^2f' + f^3g' + f^3g'}{rgf^3}$$

$$\Rightarrow R_{11} = \frac{rg^2f\ddot{g} - rg^2\dot{g}\dot{f} - rgf^2f'' + rg'f^2f' + 2f^3g'}{rgf^3}$$

$$\Rightarrow R_{10} = R_{12} = R_{13} = 0$$

$$\Rightarrow R_{22} = R^0{}_{202} + R^1{}_{212} + R^2{}_{222} + R^3{}_{232}$$

$$= -\left(\frac{f'r}{fg^2}\right) + \frac{rg'}{g^3} + \left(1 - \left(\frac{1}{g^2}\right)\right)$$

$$\Rightarrow R_{22} = \frac{-grf' + rg'f + fg^3 - fg}{fg^3}$$

$$\Rightarrow R_{20} = R_{21} = R_{23} = 0$$

$$\Rightarrow R_{33} = R^0{}_{303} + R^1{}_{313} + R^2{}_{323} + R^3{}_{333}$$

$$= -\frac{f'rsin^2\theta}{fg^2} + \frac{rg'sin^2\theta}{g^3} + sin^2\theta\left(1 - \frac{1}{g^2}\right)$$

$$\Rightarrow R_{33} = \frac{-gf'rsin^2\theta + rg'fsin^2\theta + fg^3sin^2\theta - fgsin^2\theta}{fg^3}$$

$$\Rightarrow R_{30} = R_{31} = R_{32} = 0$$

The results are summarized in the table 3.8.

Step 5:

Solving the differential equations;

We start by equating each Ricci tensor term to zero as follows:

$$R_{00} = \frac{rgf^2f'' - rg'f^2f' - rg^2f\ddot{g} + rg^2\dot{g}\dot{f} + 2gf^2f'}{g^3fr} = 0$$

$$\Rightarrow rgf^2f'' - rg'f^2f' - rg^2f\ddot{g} + rg^2\dot{g}\dot{f} + 2gf^2f' = 0 \qquad (122)$$

Dividing both sides by rfg we get:

$$ff'' - \frac{g'ff'}{g} - g\ddot{g} + \frac{g\dot{g}\dot{f}}{f} + \frac{2ff'}{r} = 0 \qquad (123)$$

$$R_{11} = \frac{rg^2f\ddot{g} - rg^2\dot{g}\dot{f} - rgf^2f'' + rg'f^2f' + 2f^3g'}{rgf^3} = 0$$

$$\Rightarrow rg^2f\ddot{g} - rg^2\dot{g}\dot{f} - rgf^2f'' + rg'f^2f' + 2f^3g' = 0$$

Dividing both sides by rfg also we get:

$$\Rightarrow g\ddot{g} - \frac{g\dot{g}\dot{f}}{f} - ff'' + \frac{g'ff'}{g} + \frac{2f^2g'}{rg} = 0 \qquad (124)$$

$$R_{22} = \frac{-grf' + rg'f + fg^3 - fg}{fg^3} = 0$$

$$\Rightarrow -grf' + rg'f + fg^3 - fg = 0 \qquad (125)$$

$$R_{33} = \frac{-gf'rsin^2\theta + rg'fsin^2\theta + fg^3sin^2\theta - fgsin^2\theta}{fg^3} = 0$$

$$\Rightarrow -gf'rsin^2\theta + rg'fsin^2\theta + fg^3sin^2\theta - fgsin^2\theta = 0$$

Dividing both sides by $sin^2\theta$

$$\Rightarrow -gf'r + rg'f + fg^3 - fg = 0 \qquad (126)$$

So we have equations (125) and (126) to be equal and this reduces the equations to three, two of which are linearly dependent. Hence they can be easily solved as follows:

$$ff'' - \frac{g'ff'}{g} - g\ddot{g} + \frac{g\dot{g}\dot{f}}{f} + \frac{2ff'}{r} = 0 \qquad (123)$$

$$g\ddot{g} - \frac{g\dot{g}\dot{f}}{f} - ff'' + \frac{g'ff'}{g} + \frac{2f^2g'}{rg} = 0 \qquad (124)$$

$$-gf'r + rg'f + fg^3 - fg = 0 \qquad (125)$$

$$(83) + (84) \Rightarrow \frac{f'}{f} + \frac{g'}{g} = 0 \Rightarrow \frac{f'}{f} = -\frac{g'}{g}$$

$$\Rightarrow \frac{df}{dr}.\frac{1}{f} = -\frac{1}{g}.\frac{dg}{dr} \Rightarrow \frac{df}{f} = -\frac{dg}{g}$$

Therefore, integrating both sides we get:

$$\int \frac{df}{f} = \int -\frac{dg}{g} \Rightarrow \ln f + \ln c = -\ln g \;\; or \ln(fgc) = 0$$

$$\Rightarrow fgc(t) = 1 \; or \; g = \frac{1}{c(t)f} \qquad (127)$$

Therefore putting (127) into (125), we get:

$$-gf'r + rg'f + fg^3 - fg = 0$$

$$\Rightarrow -\left(\frac{1}{c(t)f}\right)f'r + rf\frac{d}{dr}\left(\frac{1}{c(t)f}\right) + f\left(\frac{1}{c(t)f}\right)^3 - f\left(\frac{1}{c(t)f}\right) = 0$$

$$\Rightarrow -\left(\frac{1}{c(t)f}\right)f'r + rf\left(-\frac{1}{c(t)f^2}\right)f' + f\left(\frac{1}{c(t)f}\right)^3 - f\left(\frac{1}{c(t)f}\right) = 0$$

$$\Rightarrow -\left(\frac{1}{c(t)f}\right)f'r - \left(\frac{1}{c(t)f}\right)f'r + \left(\frac{1}{[c(t)]^3f^2}\right) - \frac{1}{c(t)} = 0$$

$$\Rightarrow -2\left(\frac{1}{c(t)f}\right)f'r + \left(\frac{1}{[c(t)]^3f^2}\right) - \frac{1}{c(t)} = 0$$

Multiplying through by c(t) leads to:

$$-2\left(\frac{1}{f}\right)f'r+\left(\frac{1}{[c(t)]^2f^2}\right)-1=0$$

Dividing through both sides by -2r also leads to:

$$\left(\frac{f'}{f}\right)-\left(\frac{1}{2r[c(t)]^2f^2}\right)+\frac{1}{2r}=0$$

$$\Rightarrow\frac{f'}{f}=\frac{1}{2r}\left(\frac{1}{[c(t)]^2f^2}-1\right)=\frac{1}{2r}\left(\frac{1-c^2f^2}{[c(t)]^2f^2}\right)$$

Separating variables leads to:

$$\Rightarrow\frac{1}{2r}=\frac{1}{f}\frac{df}{dr}\left(\frac{[c(t)]^2f^2}{1-[c(t)]^2f^2}\right)$$

and integrating both sides leads to:

$$\int\frac{1}{2r}dr=\int\frac{1}{f}\left(\frac{[c(t)]^2f^2}{1-[c(t)]^2f^2}\right)df$$

$$\Rightarrow\frac{1}{2}(\ln r+\ln c'(t))=\int\left(\frac{[c(t)]^2f}{1-[c(t)]^2f^2}\right)df=-\frac{1}{2}\ln(1-[c(t)]^2f^2)$$

$$\Rightarrow\frac{1}{2}(\ln r+\ln c')+\frac{1}{2}\ln(1-[c(t)]^2f^2)=0$$

$$\Rightarrow\ln rc'(t)\,(1-[c(t)]^2f^2)=0\Rightarrow rc'(t)(1-[c(t)]^2f^2)=1$$

$$\Rightarrow(1-[c(t)]^2f^2)=\frac{1}{rc'(t)}$$

$$\Rightarrow f = \frac{1}{c(t)}\sqrt{1 - \frac{1}{rc'(t)}} \tag{128}$$

Therefore, putting (128) into (127) we get

$$g = \frac{1}{c(t)f} = \frac{1}{c(t)\left(\frac{1}{c(t)}\sqrt{1 - \frac{1}{rc'(t)}}\right)} = \frac{1}{\sqrt{1 - \frac{1}{rc'(t)}}} \tag{129}$$

Now putting $(128) and\ (129) into\ (123)$

$$ff'' - \frac{g'ff'}{g} - g\ddot{g} + \frac{g\dot{g}\dot{f}}{f} + \frac{2ff'}{r} = 0 \tag{123}$$

But we see that part of (123) is (102).

$$ff'' - \frac{g'ff'}{g} + \frac{2ff'}{r} = f'' - \frac{g'f'}{g} + \frac{2f'}{r} = 0 \tag{130}$$

Now let

$$\frac{1}{c(t)} = k(t) and\ \frac{1}{c'(t)} = h(t) \tag{131}$$

$$\Rightarrow f = \frac{1}{c(t)}\sqrt{1 - \frac{1}{rc'(t)}} = k(t)\sqrt{1 - \frac{h(t)}{r}} \tag{132}$$

and

$$g = \frac{1}{\sqrt{1 - \frac{1}{rc'(t)}}} = \frac{1}{\sqrt{1 - \frac{h(t)}{r}}} \tag{133}$$

Therefore, putting $(132) and\ (133) into\ (130)$, we get:

$$f'' - \frac{g'f'}{g} + \frac{2f'}{r} = k(t)\frac{d}{dr}\left(\frac{1}{2}\left[\sqrt{1-\frac{h(t)}{r}}\right]^{-\frac{1}{2}}\left[\frac{h(t)}{r^2}\right]\right)$$

$$-k(t)\left(\sqrt{1-\frac{h(t)}{r}}\right)\left(-\frac{1}{2}\left[\sqrt{1-\frac{h(t)}{r}}\right]^{-\frac{3}{2}}\left[\frac{h(t)}{r^2}\right]\right)\left(\frac{1}{2}\left[\sqrt{1-\frac{h(t)}{r}}\right]^{-\frac{1}{2}}\left[\frac{h(t)}{r^2}\right]\right)$$

$$+\frac{2k(t)}{r}\left(\frac{1}{2}\left[\sqrt{1-\frac{h(t)}{r}}\right]^{-\frac{1}{2}}\left[\frac{h(t)}{r^2}\right]\right)$$

$$=\frac{k(t)}{2}\left(-\frac{1}{2}\left[\sqrt{1-\frac{h(t)}{r}}\right]^{-\frac{3}{2}}\left[\frac{h(t)}{r^2}\right]^2+\left[\sqrt{1-\frac{h(t)}{r}}\right]^{-\frac{1}{2}}\left[-2\frac{h(t)}{r^3}\right]\right)$$

$$+\frac{k(t)}{4}\left(\left[\sqrt{1-\frac{h(t)}{r}}\right]^{-\frac{3}{2}}\left[\frac{h(t)}{r^2}\right]^2\right)+\frac{2k(t)}{r}\left(\frac{1}{2}\left[\sqrt{1-\frac{h(t)}{r}}\right]^{-\frac{1}{2}}\left[\frac{h(t)}{r^2}\right]\right)$$

$$=-\frac{k(t)}{4}\left(\left[\sqrt{1-\frac{h(t)}{r}}\right]^{-\frac{3}{2}}\left[\frac{h(t)}{r^2}\right]^2\right)+\frac{k(t)}{4}\left(\left[\sqrt{1-\frac{h(t)}{r}}\right]^{-\frac{3}{2}}\left[\frac{h(t)}{r^2}\right]^2\right)$$

$$+\frac{2k(t)}{r}\left(\frac{1}{2}\left[\sqrt{1-\frac{h(t)}{r}}\right]^{-\frac{1}{2}}\left[\frac{h(t)}{r^2}\right]\right)-\frac{2k(t)}{r}\left(\frac{1}{2}\left[\sqrt{1-\frac{h(t)}{r}}\right]^{-\frac{1}{2}}\left[\frac{h(t)}{r^2}\right]\right)=0$$

Now we solve the remaining part of (123)

$$-g\ddot{g}+\frac{g\dot{g}\dot{f}}{f}=0 \Rightarrow \ddot{g}=\frac{\dot{f}}{f}$$

Therefore, separating variables and integrating both sides,

$$\int\frac{\ddot{g}}{\dot{g}}=\int\frac{\dot{f}}{f} \Rightarrow lnf(t)+ln\alpha(t)=ln\dot{g}$$

$$\Rightarrow \dot{g}=f(t)\alpha(t)$$

$$\Rightarrow -\frac{1}{2}\left[\frac{1}{\left(1-\frac{h(t)}{r}\right)^{\frac{3}{2}}}\right]h(\dot{t})=\alpha(t)k(t)\sqrt{1-\frac{h(t)}{r}}$$

$$\Rightarrow \alpha(t)k(t)=-\frac{1}{2}\left[\frac{1}{\left(1-\frac{h(t)}{r}\right)^{2}}\right]h(\dot{t})$$

Since $\alpha(t)k(t)$ is a function of t alone,

$$\therefore \frac{d}{dr}\alpha(t)k(t)=0$$

$$\Rightarrow \frac{d}{dr}\alpha(t)k(t)=\frac{d}{dr}\left\{-\frac{1}{2}\left[\frac{1}{\left(1-\frac{h(t)}{r}\right)^{2}}\right]h(\dot{t})\right\}=0$$

But since

$$\frac{d}{dr}\left\{-\frac{1}{2}\left[\frac{1}{\left(1-\frac{h(t)}{r}\right)^2}\right]\right\} \neq 0 \therefore \dot{h(t)} = 0$$

$\Longrightarrow h(t)$ is a constant $= 2GM$(say)

$$\Rightarrow f = k(t)\sqrt{1-\frac{h(t)}{r}} = k(t)\sqrt{1-\frac{2GM}{r}} \tag{134}$$

and

$$g = \frac{1}{\sqrt{1-\frac{2GM}{r}}} \tag{135}$$

Therefore, putting (134) and (135) into our metric form in (113),

$$ds^2 = -\left(k(t)\sqrt{1-\frac{2GM}{r}}\right)^2 dt^2 + \left(\frac{1}{\sqrt{1-\frac{2GM}{r}}}\right)^2 dr^2 + r^2 d\Omega^2$$

$$= -\left(\sqrt{1-\frac{2GM}{r}}\right)^2 k^2(t)dt^2 + \left(\frac{1}{\sqrt{1-\frac{2GM}{r}}}\right)^2 dr^2 + r^2 d\Omega^2$$

$$\Rightarrow ds^2 = -\left(1-\frac{2GM}{r}\right)k^2(t)dt^2 + \frac{dr^2}{\left(1-\frac{2GM}{r}\right)} + r^2 d\Omega^2 \tag{136}$$

Let $dt' = k(t)dt$

$$\Rightarrow ds^2 = -\left(1 - \frac{2GM}{r}\right)dt'^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2 d\Omega^2 \tag{137}$$

The equality of (137) and (112) proves the Birkhoff's theorem.

VITAE

UKASHAT MUBARAK SULAIMAN O.

NO. 1 AKPA QUARTERS, IYAKPI SOUTH IBIE, ETSAKO WEST L.G.A EDO STATE
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CAREER OBJECTIVES:

To strive for excellence and precision at all times in all positions and circumstances, attaining professional distinction and proficiency and achieving managerial goals.

PERSONAL DATA:

SEX:	Male
DATE OF BIRTH:	20th march 1987
LOCAL GOVT. AREA:	Etsako west
STATE OF ORIGIN:	Edo State
MARITAL STATUS:	Single
NATIONALITY:	Nigerian
RELIGION:	Islam
MAJOR:	Physics and Petroleum Engineering
PERMAMENT HOME ADDRESS:	No. 1 Akpa quarters, South Ibie, Iyakpi, Etsako West L.G.A. Edo State. Nigeria
CONTACT ADDRESS:	No. 7 Ojokoh lane off old Benin-Auchi road. P.O.Box 930, Ikpoba hill Urora quarters, Benin city, Edo State. Nigeria
PHONE NUMBER:	+14352372272, +966508364295, +2348065244000
COMPLETION DATE OF NYSC:	2013

SCHOOLS ATTENDED WITH DATES

SCHOOLS	FROM	TO
Utah State University, Logan UT84321, USA	2015	Date
King Fahd University of Petroleum and Minerals Dhahran Saudi Arabia	2013	2016

Federal University of Petroleum Resources,Effurun, Delta State Nigeria	2007	2012
Petroleum Training Institue,Effurun, Delta State Nigeria	2008	2013
Islamic Centre AnofiaAfikpo,Ebonyi State, Nigeria	1998	2004
Adolor Primary School Benin City, Edo State, Nigeria	1993	1998

QUALIFICATIONS OBTAINED

PhD. Particle Astrophysics and Cosmology	In Progress
M.Sc. Physics CGPA(3.63/4)	2016
B.Sc. Physics CGPA (4.29/5)	2012
National Diploma in Petroleum Engineering From Petroleum Training Institute Effurun Delta State CGPA (3.39/4)	2013
West African Senior School Certificate Examination (WASSCE)	2004

WORKING EXPERIENCE

EMPLOYER AND POSITION HELD	FROM	TO
Henson Demonstration College Benin City, Nigeria (physics teacher)	**2005**	**2007**
Nigerian Petroleum Development Company Benin City, Nigeria (Trainee under the petroleum engineering Unit). Worked at the flow station and with the Driller in a field operation	August 2008	November 2008
Ochuks Academy Delta Palace Road Ogborikoko, Delta State, Nigeria (physics teacher)	August 2010	January 2011

Centre of Excellence in Nanotechnology and petrophysics, King Fahd University. Research Assistant Trainee: to operate various characterization and imagingequipment involving - Computed Tomography for analyzing core samples in 3D - Thermal Analysis Technique - X-ray Diffraction analysis - Neutron Activation Analysis - Prompt Gamma Neutron Activation Analysis - Crystallography and - Synthesis of nanoparticles.	2013	2015
Department of Physics, King Fahd University of Petroleum and Minerals. (Undergraduate Lab Instructor)	2014	2015
Research Teaching Assistant Utah State University USA	2015	Date

PUBLICATIONS

Pulse height tests of a large diameter fast labr3:ce scintillation detector
article in applied radiation and isotopes · august 2015

PROFESSIONAL BODIES

A member of Nigerian Association of Petroleum Explorationist (NAPE)

A member of Society for Petroleum Engineers (SPE)

HOBBIES

Reading, Research and sporting activities

LANGUAGES UNDERSTOOD

English, Arabic, a bit of German

REFEREES

1.) **H.R.H AlhajiAliyu K. Danesi**

The Aidonogie of South Ibie,

Etsako West L.G.A Edo State. Nigeria

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2.) **Dr. Edison Enaibe**

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3.) **Dr. James Thomas Wheeler**

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4.) **Professor Ashfaque Hussain Bokari**

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5.) **Professor Thamer Al-Aithan**

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6.) **Professor Saeed Alamoudi**

Director of Academic Planning,

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7.) **Professor Abdallah Zain Yamani**

Director Center for Excellence in Nanotechnology,

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